

SUPERMANN: A SUPERLINEARLY CONVERGENT ALGORITHM FOR FINDING FIXED POINTS OF NONEXPANSIVE OPERATORS

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ABSTRACT. We propose *SuperMann*, a Newton-type algorithm for finding fixed points of nonexpansive operators. *SuperMann* generalizes the classical Krasnosel'skiĭ-Mann scheme (KM), enjoys its favourable global convergence properties and requires exactly the same oracle. The basic idea behind *SuperMann* is the interpretation of KM iterations as projections on a halfspace that separates the current iterate from the set of fixed points. Taking this idea one step further we show how to include steps along arbitrary directions in our scheme. In particular, when the directions satisfy a Dennis-Moré condition we show that *SuperMann* converges superlinearly under mild assumptions. As a result, *SuperMann* enhances and robustifies almost all operator splitting schemes for structured convex optimization, overcoming their well known sensitivity to ill-conditioning.

1. INTRODUCTION

The main motivation for this work is to eliminate the frustrating effect of slow convergence and sensitivity to parameter selection of operator splitting methods. Almost all operator splitting methods for finding a zero of the sum of monotone operators can be seen as an application of relaxed fixed-point iterations to some nonexpansive mapping related to the corresponding splitting scheme. Famous and widely implemented such instances include the forward-backward splitting (FBS), also known as proximal gradient method in convex minimization problems, the Douglas-Rachford splitting (DRS) and its dual version, the alternating direction of minimizers method (ADMM), and many others. Although sometimes a fast convergence rate can be observed, the norm of fixed-point residual decreases, at best, with Q -linear rate; moreover, due to an inherent sensitivity to ill-conditioning, oftentimes the Q -factor is close to one.

As an attempt to solve the issue, people have considered the employment of variable metrics [4] to reshape the geometry of the problem and enhance convergence rate. However, performing proximal steps in arbitrary metrics is challenging even for elementary functions, and as a result the range of metrics which are implementable in practice is considerably limited, which makes this approach, so far at least, not very appealing.

Recently, [12] observed that in many problems subsequent fixed-point iterations proceed along almost parallel directions, and proposed a line-search method that exploits this property. Step-sizes larger than the classical ones are accepted as long as the residual of the candidate next point is sufficiently smaller than that of the

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current iterate. Though this heuristic in practice leads to faster convergence, however it neither improves the theoretical convergence rates nor can it cope with ill-conditioning.

Alternatively, the task of searching fixed points of an operator T can be translated to that of finding zeros of the corresponding residual $R = \text{Id} - T$. Many methods with fast asymptotic convergence rates such as Newton-type exist that can be employed for efficiently solving nonlinear equations, see *e.g.* [17, §11], [11, §7] and [15]. However, such methods converge only when close enough to the solution, and in order to globalize the convergence there comes the need of a merit function to perform line-searches along candidate directions of descent. The typical and universal choice of the square residual $\|Rx\|^2$ unfortunately is of little use, as in meaningful applications the mapping R is nonsmooth.

The recently introduced *proximal envelopes* [20, 21] provide a positive answer to this issue by serving as smooth merit functions for FBS, DRS and ADMM applied to composite convex minimization problems. Quasi-Newton methods are proven to enhance such splitting schemes yielding asymptotic superlinear convergence rates also for nonconvex problems [26, 27]. The efficacy of the approach is also supported by extensive numerical simulations, and the forward-backward-envelope-based ForBES solver is available online.¹

Though proximal envelopes are extremely valuable tools whose properties and employment surely deserve further investigation, which indeed is the purpose of the generalized framework [13], they cannot handle, for instance, saddle-point convex-concave optimization problems typically arising from primal-dual splittings such as Vñ-Condat [5].

In response to this issue, in this paper we propose a universal scheme that globalizes Newton-type methods for finding fixed points of arbitrary nonexpansive operators. Our idea is based on an elementary geometrical interpretation of the classical Krasnosel'skiĭ-Mann (KM) iterations thanks to which we are able to move along arbitrary *fast* directions, yet preserving (quasi-) Fejér monotonicity of the KM scheme and by performing solely evaluations of T . We propose a globally convergent method that, relative to the classical KM scheme, (1) allows for the integration of any (fast) update direction, (2) requires the same oracle information (evaluations of T), and (3) includes KM as a special case.

Furthermore, we consider a modified Broyden's scheme which was first introduced in [22]. We first prove its superlinear convergence in solving nonlinear equations in arbitrary Hilbert spaces, extending the results in the literature which are limited to finite-dimensional spaces [16, 6] or rely on differentiability assumptions and require the initial Broyden operator to be sufficiently close to the Jacobian at the solution [24]. Then we show how the scheme fits into our framework and enables superlinear asymptotic convergence rate of the proposed algorithm.

Admittedly with an intended pun, since it exhibits *super*linear convergence rates and generalizes the Krasnosel'skiĭ-Mann iterations we name our method *SuperMann scheme*. We develop the theory in real Hilbert spaces.

The paper is organized as follows. In Section 2 we introduce some basic notation and known facts. In Section 3 we define the problem at hand and propose a general abstract algorithmic framework for solving it, proving its convergence properties. In Section 4 we provide a generalization of the classical KM-iterations that are key for the global convergence and performance of the *SuperMann scheme*. In Section 5 we present the *SuperMann scheme*, an efficient implementation of the general algorithmic framework described in Section 3, and prove its global and local convergence properties. In Section 6 we propose the modified Broyden's scheme and prove that

¹<http://kul-forbes.github.io/ForBES/>

its integration in the *SuperMann scheme* yields superlinear convergence rates; for the sake of readability some of the proofs are referred to [Appendix A](#). Finally, in [Section 7](#) we show how the theoretical findings are backed up by promising numerical simulations, where *SuperMann scheme* dramatically improves classical splitting schemes and popular solvers.

2. BASIC NOTATION AND DEFINITIONS

In this section we introduce some notational conventions and briefly list some known results.

To streamline the notation, given a sequence $(x^k)_{k \in \mathbb{N}}$ and a set A we write $(x^k)_{k \in \mathbb{N}} \subset A$ with the obvious meaning of $x^k \in A$ for all $k \in \mathbb{N}$. For $p > 0$ we let

$$\ell^p := \{(x^k)_{k \in \mathbb{N}} \subset \mathbb{R} \mid \sum_{k \in \mathbb{N}} |x^k|^p < \infty\}$$

denote the set of real-valued sequences with summable p -th power, and with ℓ_+^p the subset made of those which are additionally positive-valued.

Given $x \in \mathbb{R}$ we let $[x]_+ := \max\{x, 0\}$ denote the positive part of x .

2.1. Hilbert spaces and bounded linear operators. Throughout the paper, \mathcal{H} is a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and with induced norm $\|\cdot\|$. For $\bar{x} \in \mathcal{H}$ and $r > 0$ we let $B(\bar{x}, r) := \{x \in \mathcal{H} \mid \|x - \bar{x}\| < r\}$ denote the open ball centered at \bar{x} with radius r .

Given $(x^k)_{k \in \mathbb{N}} \subset \mathcal{H}$ and $x \in \mathcal{H}$ we write $x^k \rightarrow x$ and $x^k \rightharpoonup x$ to denote, respectively, strong and weak convergence of $(x^k)_{k \in \mathbb{N}}$ to x , and when not specified convergence is meant in the strong sense. With $\mathcal{W}(x^k)_{k \in \mathbb{N}}$ we denote the set of weak sequential cluster points of $(x^k)_{k \in \mathbb{N}}$.

With $\mathcal{B}(\mathcal{H})$ we denote the set of bounded linear operators $\mathcal{H} \rightarrow \mathcal{H}$, and with Id the identity $x \mapsto x$. For $L \in \mathcal{B}(\mathcal{H})$ we let $\|L\|_{\mathcal{B}}$ denote the norm of L , namely,

$$\|L\|_{\mathcal{B}} := \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|} = \sup_{x: \|x\|=1} \|Lx\|.$$

L^* denotes the adjoint operator of L , *i.e.*, the unique in $\mathcal{B}(\mathcal{H})$ such that

$$\langle Lx, y \rangle = \langle x, L^*y \rangle \quad \text{for all } x, y \in \mathcal{H}$$

and we say that L is symmetric if $L = L^*$.

For $u, v \in \mathcal{H}$, with $u \otimes v \in \mathcal{B}(\mathcal{H})$ we refer to the rank-one operator

$$(u \otimes v)x := \langle v, x \rangle u$$

which can be easily shown to satisfy

$$\|u \otimes v\|_{\mathcal{B}} = \|u\| \|v\|. \quad (2.1)$$

We say that $L \in \mathcal{B}(\mathcal{H})$ is STRONGLY NONSINGULAR if

$$\inf_{x \neq 0} \frac{\|Lx\|}{\|x\|} > 0$$

which clearly implies that L is invertible, the conditions being equivalent if \mathcal{H} is finite-dimensional.

2.2. Nonexpansive operators. In this section we briefly recap some known results of nonexpansive operator theory that will be used throughout the paper.

Definition 2.1. An operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be

(i) NONEXPANSIVE (NE) if

$$\forall x, y \in \mathcal{H} \quad \|Tx - Ty\| \leq \|x - y\|$$

(ii) FIRMLY NONEXPANSIVE (FNE) if

$$\forall x, y \in \mathcal{H} \quad \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(\text{Id} - T)x - (\text{Id} - T)y\|^2$$

(iii) AVERAGED if it is α -AVERAGED for some $\alpha \in (0, 1)$, i.e., if there exists a nonexpansive operator $S : \mathcal{H} \rightarrow \mathcal{H}$ such that $T = (1 - \alpha)\text{Id} + \alpha S$.

For notational convenience we extend the definition of α -averagedness to the case $\alpha = 1$ which reduces to plain nonexpansiveness. Moreover, excluding the case $\alpha = 0$ causes no loss of generality, in that the only 0-averaged operator is the identity which however is α -averaged for any $\alpha \in (0, 1)$.

Given an operator $T : \mathcal{H} \rightarrow \mathcal{H}$ we let

$$\mathbf{zer} T := \{z \in \mathcal{H} \mid Tz = 0\}$$

denote the set of its ZEROS, and

$$\mathbf{fix} T := \{z \in \mathcal{H} \mid Tz = z\} = \mathbf{zer}(\text{Id} - T)$$

the set of its FIXED POINTS. Moreover, for $\lambda \in \mathbb{R}$ we let

$$T_\lambda := (1 - \lambda)\text{Id} + \lambda T.$$

We now list a few known results on NE operators.

Lemma 2.2 ([1, Prop. 4.2]). *Let $T : \mathcal{H} \rightarrow \mathcal{H}$. The following are equivalent:*

- (a) T is FNE;
- (b) $\text{Id} - T$ is FNE;
- (c) $\langle Tx - Ty, (\text{Id} - T)x - (\text{Id} - T)y \rangle \geq 0$ for all $x, y \in \mathcal{H}$;
- (d) T is $1/2$ -averaged.

Lemma 2.3. *Suppose $T : \mathcal{H} \rightarrow \mathcal{H}$ is α -averaged, and let $R = \text{Id} - T$. Then,*

- (i) $\mathbf{zer} T$ and $\mathbf{fix} T$ are closed and convex;
- (ii) $\text{Id} - T_\lambda = \lambda R$; in particular $\mathbf{zer} T_\lambda = \mathbf{zer} T$ and $\mathbf{fix} T_\lambda = \mathbf{fix} T$ for all $\lambda \neq 0$;
- (iii) T_λ is $\alpha\lambda$ -averaged for any $\lambda \in [0, 1/\alpha]$;
- (iv) $T_{1/2\alpha}$ is FNE;
- (v) $\|RT_\lambda x\| \leq \|Rx\|$ for any $\lambda \in [0, 1/\alpha]$;
- (vi) $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha} \|Rx - Ry\|^2$ for all $x, y \in \mathcal{H}$;
- (vii) $\mathbf{dist}(Tx, \mathbf{fix} T)^2 \leq \mathbf{dist}(x, \mathbf{fix} T)^2 - \frac{1-\alpha}{\alpha} \|Rx\|^2$ for all $x \in \mathcal{H}$.

Proof. 2.3(i) and 2.3(vi) are proven in Cor. 4.15 and Prop. 4.25 of [1], respectively, while 2.3(ii) is a straightforward computation.

For some NE operator S it holds that $T = (1 - \alpha)\text{Id} + \alpha S$, therefore

$$T_\lambda = (1 - \lambda)\text{Id} + \lambda((1 - \alpha)\text{Id} + \alpha S) = (1 - \alpha\lambda)\text{Id} + \alpha\lambda S$$

and 2.3(iii) follows; in turn, combined with Lem. 2.2(d) it implies 2.3(iv).

2.3(v) is shown in the proof of [1, Thm. 5.14(ii)], observing that 2.3(iii) ensures T_λ to be nonexpansive.

Finally, 2.3(vii) is meaningful only if $\mathbf{fix} T \neq \emptyset$, in which case 2.3(i) ensures the existence of the projection z of x onto $\mathbf{fix} T$, and the sought inequality follows by plugging $y = z$ in 2.3(vi). \square

For a closed and nonempty convex set $C \subseteq \mathcal{H}$ we let Π_C denote the projection operator on C , and for $\lambda \in [0, 2]$ we let

$$\Pi_{C,\lambda} := (1 - \lambda)\text{Id} + \lambda \Pi_C.$$

Lemma 2.4. *Let $C \subseteq \mathcal{H}$ be a nonempty closed and convex set. Then Π_C is FNE with $\text{fix } \Pi_C = C$. Moreover, for all $\lambda \in [0, 2]$ and $x, y \in \mathcal{H}$ it holds that*

$$\|\Pi_{C,\lambda} x - \Pi_{C,\lambda} y\|^2 \leq \|x - y\|^2 - \lambda(2 - \lambda)\|(\text{Id} - \Pi_C)x - (\text{Id} - \Pi_C)y\|^2. \quad (2.2)$$

In particular,

$$\|\Pi_{C,\lambda} x - z\|^2 \leq \|x - z\|^2 - \lambda(2 - \lambda)\|x - \Pi_C x\|^2 \quad \forall x \in \mathcal{H}, z \in C. \quad (2.3)$$

Proof. That Π_C is FNE is shown in [1, Prop. 4.8]. As to (2.2), if $\lambda = 0$ then $\Pi_{C,\lambda} = \text{Id}$ and the claim is trivial. Otherwise, from Lem. 2.2(d) it follows that $\Pi_{C,\lambda}$ is $\lambda/2$ -averaged; for $\lambda = 2$ (2.2) is due to nonexpansiveness of $\Pi_{C,\lambda}$, while for $\lambda \in (0, 2)$ the sought inequality follows from [1, Prop. 4.25(iii)] with $\alpha = \lambda/2 \in (0, 1)$. \square

2.3. Fejér sequences.

Definition 2.5. *Relative to a nonempty set $S \subseteq \mathcal{H}$, a sequence $(x^k)_{k \in \mathbb{N}} \subset \mathcal{H}$ is*

(i) FEJÉR (MONOTONE) *if*

$$\|x^{k+1} - s\| \leq \|x^k - s\| \quad \forall k \in \mathbb{N}, s \in S$$

(ii) QUASI-FEJÉR (MONOTONE) *if for all $s \in S$ there exists $(\varepsilon_k(s))_{k \in \mathbb{N}} \in \ell_+^1$ such that*

$$\|x^{k+1} - s\|^2 \leq \|x^k - s\|^2 + \varepsilon_k(s) \quad \forall k \in \mathbb{N}$$

Quasi-Fejér monotonicity as in Definition 2.5(ii) is a broader definition than the one in [10] where the notion was first introduced (cf. Rem. 2.6). More precisely, our concept of quasi-Fejér monotonicity is taken from [3] where is referred to as *of type III* to differentiate it from the original definition which instead is referred to as *of type I*. In this paper however we do not make this distinction and always stick to Definition 2.5(ii).

Remark 2.6. Quasi-Fejér monotonicity implies boundedness of the sequence, and therefore it is in particular implied by the nonsquared inequality

$$\|x^{k+1} - s\| \leq \|x^k - s\| + \varepsilon_k \quad \exists (\varepsilon_k)_{k \in \mathbb{N}} \in \ell_+^1$$

which corresponds to the original concept of Fejér-monotonicity [10]. \square

Theorem 2.7. *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a NE operator with $\text{fix } T \neq \emptyset$, and suppose that $(x^k)_{k \in \mathbb{N}} \subset \mathcal{H}$ is quasi-Fejér with respect to $\text{fix } T$. If $(x^k - Tx^k)_{k \in \mathbb{N}} \rightarrow 0$, then there exists $x^* \in \text{fix } T$ such that $x^k \rightharpoonup x^*$.*

Proof. From [3, Prop. 3.7(i)] we have $\mathcal{W}(x^k)_{k \in \mathbb{N}} \neq \emptyset$; in turn, from [1, Cor. 4.18] we infer that $\mathcal{W}(x^k)_{k \in \mathbb{N}} \subseteq \text{fix } T$. The claim then follows from [3, Thm. 3.8]. \square

2.4. Generalized differentiability. For the reader's convenience we list some known result for semidifferentiable functions which are proven in the literature for functions in \mathbb{R}^n , but whose proofs straightforwardly extend to Hilbert spaces. Recall that $F : \mathcal{H} \rightarrow \mathcal{H}$ is DIFFERENTIABLE at \bar{x} if there exists $JF(\bar{x}) \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{\substack{x \rightarrow \bar{x} \\ x \neq \bar{x}}} \frac{Fx - F\bar{x} - JF(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

In such case, $JF(\bar{x})$ is unique and is the JACOBIAN of F at \bar{x} . The next result follows straightforwardly from the definition of differentiability.

Lemma 2.8. *Suppose that $F : \mathcal{H} \rightarrow \mathcal{H}$ is differentiable at \bar{x} and $JF(\bar{x}) \in \mathcal{B}(\mathcal{H})$ is strongly nonsingular. Then, there exist two constants $\varepsilon, \kappa > 0$ such that*

$$\|Fx - F\bar{x}\| \geq \kappa\|x - \bar{x}\| \quad \forall x \in B(x, \varepsilon).$$

We say that $F : \mathcal{H} \rightarrow \mathcal{H}$ is CALM at $\bar{x} \in \mathcal{H}$ if there exist a neighborhood U_{x^*} of x^* and a constant $L_F \geq 0$ such that $\|Fx - F\bar{x}\| \leq L_F\|x - \bar{x}\|$ for all $x \in U_{x^*}$. Calmness is somewhere referred to as Lipschitz continuity at one point.

Definition 2.9 (Strict differentiability). *We say that $F : \mathcal{H} \rightarrow \mathcal{H}$ is STRICTLY DIFFERENTIABLE at \bar{x} if it is differentiable there with $JF(\bar{x})$ satisfying*

$$\lim_{\substack{(y,z) \rightarrow (\bar{x}, \bar{x}) \\ y \neq z}} \frac{\|Fy - Fx - JF(\bar{x})(y - x)\|}{\|y - x\|} = 0. \quad (2.4)$$

There is a slight ambiguity in the literature, whereas such property is sometimes referred to as *strong* (see e.g. [19, 14]) rather than *strict* differentiability. We stick to the proposed notation which is taken from [23].

Definition 2.10 (Semiderivative). *We say that an operator $F : \mathcal{H} \rightarrow \mathcal{H}$ is SEMIDIFFERENTIABLE at \bar{x} if there exists a continuous and positively homogeneous function $DF(\bar{x}) : \mathcal{H} \rightarrow \mathcal{H}$, called the SEMIDERIVATIVE of F at \bar{x} , such that*

$$Fx = F\bar{x} + DF(\bar{x})[x - \bar{x}] + o(\|x - \bar{x}\|).$$

We say that the semiderivative is CONTINUOUS (resp. (CALM)) at \bar{x} if the function $x \mapsto DF(x)[w]$ is continuous (resp. calm) at \bar{x} for all $w \in \mathcal{H}$. STRICT SEMIDIFFERENTIABILITY is defined as in Definition 2.9.

By positive homogeneity, continuity and/or calmness of $x \mapsto DF(x)[w]$ for all $w \in \mathcal{H}$ is equivalent to that for all $w \in \mathcal{H}$ with $\|w\| = 1$. Semidifferentiability is clearly a milder property than differentiability in that the mapping $DF(\bar{x})$ needs not be linear; the interested reader is referred to [23, §7] for an extensive discussion. When F is strictly continuous (i.e., locally Lipschitz-continuous) at \bar{x} , semidifferentiability at \bar{x} is equivalent to directional differentiability at \bar{x} [11, Prop. 3.1.3] and the semiderivative is sometimes rather called *B-derivative* (see e.g. [19, 14, 11]).

Theorem 2.11 ([19, Thm. 2]). *Let $U \subseteq \mathcal{H}$ be a nonempty open set and let $\bar{x} \in U$. Suppose that $F : U \rightarrow \mathcal{H}$ is Lipschitz-continuous and semidifferentiable. Then, the following statements are equivalent:*

- (a) *F is strictly differentiable at \bar{x} ;*
- (b) *F is strictly semidifferentiable at \bar{x} ;*
- (c) *the semiderivative $DF(\bar{x})$ is continuous at \bar{x} .*

Lemma 2.12. [14, Lem. 2.2] *Let $U \subseteq \mathcal{H}$ be a nonempty open set and let $\bar{x} \in U$. Suppose that $F : U \rightarrow \mathcal{H}$ is Lipschitz continuous on U and L_F -calmly semidifferentiable at \bar{x} . Then, there exists a neighborhood $U_{\bar{x}}$ of \bar{x} such that*

$$\|Fx - Fy - DF(\bar{x})[x - y]\| \leq L_F \max\{\|x - \bar{x}\|, \|y - \bar{x}\|\} \|x - y\| \quad \forall x, y \in U_{\bar{x}}.$$

3. GENERAL ABSTRACT FRAMEWORK

In the rest of the paper we work under the following assumption.

Assumption I. *$T : \mathcal{H} \rightarrow \mathcal{H}$ is an α -averaged operator for some $\alpha \in (0, 1]$ and with $\mathbf{fix} T \neq \emptyset$. With $R := \text{Id} - T$ we denote its (2α -Lipschitz continuous) fixed-point residual.*

Given T and R as in Assumption I, our goal is to find a fixed point of T , or, equivalently, a zero of R :

$$\text{find } x^* \in \mathbf{fix} T = \mathbf{zer} R. \quad (3.1)$$

The assumption of nonemptiness of $\mathbf{fix} T$ is then equivalent to requiring that the proposed problem has a solution.

In this section we introduce an abstract procedure to solve problem (3.1). The scheme is not implementable in and of itself, as it gives no hint as to how to compute each of the iterates, but it rather serves as a comprehensive ground framework for a class of algorithms with global convergence guarantees. In Section 5 we will

derive the *SuperMann scheme*, a concrete implementable instance which also enjoys appealing asymptotic properties when *good* update directions are selected.

The general framework prescribes updates that fall into three categories.

- K_0) **Blind updates.** Inspired from [2], whenever the residual $\|Rx^k\|$ at iteration k has *sufficiently* decreased with respect to past iterates we allow for an *uncontrolled* update; namely, when the current situation is particularly favorable we allow for an (almost) arbitrary x^{k+1} . For an efficient implementation such guess should be somehow reasonable and not completely a *blind* leap of faith; however, for the sake of global convergence the proposed scheme is robust to (almost) any choice.
- K_1) **Educated updates.** The downside of a K_0 -update is that it is indeed a *blind guess*, in that it does not take into account or check the *goodness* of the update x^{k+1} . To encourage favorable updates we accept an *educated guess* x^{k+1} whenever the candidate residual satisfies $\|Rx^{k+1}\| \leq \hat{c}\|Rx^k\|$, where $\hat{c} \in (0, 1)$ is a user-defined constant (see *e.g.* [15, §5.3.1] or [11, §8.3.2]).
- K_2) **Safeguard (quasi-Fejér) updates.** This last kind of updates is similar to K_1 as it is also based on the goodness of x^{k+1} with respect to x^k . The difference is that instead of checking the residual, what needs to be *sufficiently* decreased is the distance from each point in $\text{fix } T$. This is meant in a quasi-Fejér fashion as in Definition 2.5(ii).

K_0 and K_1 are somehow complementary: the former is activated when small *perturbations* are possible thanks to the progress made so far yet no glance is taken to the future, while the latter only compares the candidate update with the current situation completely disregarding previous history. Though indeed *educated*, this second kind of updates is more risky than the first, and indeed it can be activated only when the current residual is not (too) larger than a safeguard parameter r_{safe} . While K_0 -updates are well-behaving with the other two, the safeguard parameter r_{safe} is needed to prevent things from possibly going wrong when passing from K_2 -to K_1 -updates.

Intuitively, *safeguard* updates K_2 are the ones that ensure global convergence, while *blind* and *educated* updates K_0 and K_1 are in charge of the asymptotic behavior.

To establish a notation, we partition the set of iteration indices $K \subseteq \mathbb{N}$ as $K_0 \cup K_1 \cup K_2$. Namely, relative to Algorithm 1, K_0 , K_1 and K_2 denote the sets of indices k passing the test at steps 2, 3(a) and 3(b), respectively. To rule out trivialities, throughout the paper we work under the assumption that a solution is not found in a finite number of steps, so that the residual of each iterate is always nonzero. As long as it is well defined, the algorithm therefore produces an infinite number of iterates.

More can be said about the convergence rates if the mapping R possesses METRIC SUBREGULARITY. Metric subregularity of a (possibly multivalued) operator R at \bar{x} is equivalent to calmness of the operator R^{-1} at $R\bar{x}$ [7, Thm 3.2], and is a weaker condition than metric regularity and Aubin property. We refer the reader to [23, §9] and [8, §3] for an extensive discussion. Below, we propose a simplified definition for single-valued operators and with respect to its zeros.

Definition 3.1 (Metric subregularity at zeros). *Let $R : \mathcal{H} \rightarrow \mathcal{H}$ and $\bar{x} \in \text{zer } R$. R is METRICALLY SUBREGULAR at \bar{x} if there exist $\varepsilon, \kappa > 0$ such that*

$$\text{dist}(x, \text{zer } R) \leq \kappa \|Rx\| \quad \forall x \in B(\bar{x}, \varepsilon). \quad (3.2)$$

κ and ε are (one) MODULUS and (one) RADIUS of subregularity at \bar{x} , respectively.

Theorem 3.2 (Global convergence of the general framework Algorithm 1). *Let T , R and α be as in Assumption I. Consider the iterates generated by the Algorithm 1*

Algorithm 1 *General framework* for solving (3.1), given an α -averaged operator T with residual $R = \text{Id} - T$

REQUIRE $x^0 \in \mathcal{H}$, $c, \hat{c} \in [0, 1)$, $\sigma > 0$

INITIALIZE $\eta_0 = r_{\text{safe}} = \|Rx^0\|$, $k = 0$

1. IF $Rx^k = 0$, THEN STOP.
 2. IF $\|Rx^k\| \leq c\eta_k$, THEN set $\eta_{k+1} = \|Rx^k\|$, proceed with a *blind update* x^{k+1} and go to step 4.
 3. Set $\eta_{k+1} = \eta_k$ and select x^{k+1} such that
 - 3(a) EITHER the *safe condition* $\|Rx^k\| \leq r_{\text{safe}}$ holds, and x^{k+1} is *educated*:

$$\|Rx^{k+1}\| \leq \hat{c}\|Rx^k\|$$
 - 3(b) OR it is (*quasi*-)Fejér with respect to $\mathbf{fix} T$:

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \sigma\|Rx^k\|^2 \quad \forall z \in \mathbf{fix} T. \quad (3.3)$$
- IF x^k was not computed with another *quasi-Fejér* update, THEN update $r_{\text{safe}} = \|Rx^k\| + \hat{\varepsilon}_k$ for some $\hat{\varepsilon}_k \geq 0$.
4. Set $k \leftarrow k + 1$ and go to step 1.
-

and suppose that for all k it is always possible to find a point x^{k+1} complying with the requirements of either step 2 3(a) or 3(b), and further satisfying

$$\|x^{k+1} - x^k\| \leq D\|Rx^k\| \quad (3.4)$$

for some constant $D \geq 0$. If $(\hat{\varepsilon}_k)_{k \in \mathbb{N}} \in \ell_+^1$, then

- (i) $(x^k)_{k \in \mathbb{N}}$ is *quasi-Fejér monotone* with respect to $\mathbf{fix} T$;
- (ii) $Rx^k \rightarrow 0$ with $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^2$;
- (iii) $(x^k)_{k \in \mathbb{N}}$ converges weakly to a point $x^* \in \mathbf{fix} T$;
- (iv) if $c > 0$, then the number of *blind updates* at step 2 is infinite.
- (v) If $x^k \rightarrow x^*$ (this being true if \mathcal{H} is finite-dimensional) and R is *metrically subregular* at x^* , then $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^1$.

Proof.

♠ 3.2(i): we start observing that because of the assumptions we have

$$\|x^{k+1} - z\| \leq \|x^k - z\| + D\|Rx^k\| \quad \forall z \in \mathbf{fix} T, k \in \mathbb{N} \quad (3.5)$$

and since R is 2α -Lipschitz continuous we also have that

$$\|Rx^{k+1}\| \leq \|Rx^k\| + \|Rx^{k+1} - Rx^k\| \leq (1 + 2\alpha D)\|Rx^k\|. \quad (3.6)$$

(3.5) implies that in order to prove *quasi-Fejér* monotonicity it suffices to show that $(\|Rx^k\|)_{k \in K_0 \cup K_1}$ is summable. Let the *blind update* index set K_0 be indexed as

$$K_0 = \{k_1, k_2, \dots\}.$$

Since η_k is kept constant whenever $k \notin K_0$, it holds that

$$\eta_{k_\ell} = \|Rx^{k_\ell-1}\| \leq c\eta_{k_\ell-1} \leq \dots \leq c^{\ell-1}\eta_{k_1} = c^{\ell-1}\eta_0 \quad \forall k_\ell \in K_0. \quad (3.7)$$

In particular, $(\|Rx^{k_\ell}\|)_{k_\ell \in K_0}$ is summable (regardless if K_0 is finite or not), and it only remains to prove that the sequence of residuals $(\|Rx^k\|)_{k \in K_1}$ of *educated* updates is summable. For $k \in K_1$, let

$$\kappa_1(k) := \max \{i \in K_1 \mid i \leq k, i-1 \notin K_1\} \quad (3.8)$$

be the minimum index of K_1 such that $\kappa_1(k), \kappa_1(k) + 1, \dots, k \in K_1$. Then, in light of the property $\|Rx^{k+1}\| \leq \hat{c}\|Rx^k\|$ that characterizes $k \in K_1$ it follows that

$$\|Rx^k\| = \|Rx^{k-1+1}\| \leq \hat{c}\|Rx^{k-1}\| \leq \dots \leq \hat{c}^{k-\kappa_1(k)}\|Rx^{\kappa_1(k)}\| \quad k \in K_1 \quad (3.9)$$

where we let $0^0 = 1$ by convention, so that the inequality is well defined also for $\widehat{c} = 0$ and $k = \kappa_1(k)$ (i.e., when $k-1 \notin K_1$). If $\kappa_1(k)-1 \in K_2$, then we may exploit the *safe condition* at step **3(a)** which ensures that $\|Rx^{\kappa_1(k)}\| \leq r_{\text{safe}}$. Because of how r_{safe} is updated (cf. step **3(b)**), we have $r_{\text{safe}} = \|Rx^{\widetilde{k}}\| + \widehat{\varepsilon}_{\widetilde{k}}$, where

$$\widetilde{k} = \mathbf{max} \{i \in K_2 \mid i \leq \kappa_1(k) - 1, i - 1 \notin K_2\}$$

(in analogy with the definition of (3.8) we may write $\widetilde{\kappa} = \kappa_2(\kappa_1(k) - 1) = \kappa_2(k)$). Therefore,

$$\|Rx^k\| \leq \widehat{c}^{k-\kappa_1(k)} \|Rx^{\kappa_1(k)}\| \leq \widehat{c}^{k-\kappa_1(k)} (\|Rx^{\widetilde{k}}\| + \widehat{\varepsilon}_{\widetilde{k}}) \leq \widehat{c}^{k-\kappa_1(k)} \|Rx^{\widetilde{k}}\| + \widehat{\varepsilon}_k$$

If $\widetilde{k} - 1 \in K_1$, then we may continue the chain of inequalities

$$\leq \widehat{c}^{k-\kappa_1(k)+1} \|Rx^{\widetilde{k}-1}\| + \widehat{\varepsilon}_k \quad \dots \quad (3.10)$$

and reiterate the reasoning up to when an index $k_\ell \in K_0$ is encountered. Letting

$$\kappa_0(k) := \mathbf{max} \{i \in K_0 \cup \{0\} \mid i \leq k\} \quad \text{and} \quad i_k := \#\{k' \in K_1 \mid \kappa_0(k) \leq k' < k\},$$

because of (3.6) we then have

$$\begin{aligned} \sum_{k \in K_1} \|Rx^k\| &\leq (1 + 2\alpha D) \sum_{k \in K_1} \widehat{c}^{i_k} \|Rx^{\kappa_0(k)}\| + \sum_{k \in \mathbb{N}} \widehat{\varepsilon}_k \\ &\leq (1 + 2\alpha D) \sum_{i \in \mathbb{N}} \widehat{c}^i \sum_{k \in K_0} \|Rx^k\| + \sum_{k \in \mathbb{N}} \widehat{\varepsilon}_k < \infty \end{aligned} \quad (3.11)$$

where the last inequality is due to the already proven fact that $(\|Rx^k\|)_{k \in K_0}$ is summable.

♠ **3.2(ii)**: quasi-Fejér monotonicity implies that the sequence $(x^k)_{k \in \mathbb{N}}$ is bounded, and in particular $\nu(z) := \sup(\|x^k - z\|)_{k \in \mathbb{N}} < \infty$ for all $z \in \mathcal{H}$. As shown in the previous point, $(\|Rx^k\|)_{k \in K_0 \cup K_1}$ is summable and in particular converges to 0. Moreover,

$$\begin{aligned} \|x^{k+1} - z\|^2 &\leq \|x^k - z\|^2 + D^2 \|Rx^k\|^2 + 2D \|x^k - z\| \|Rx^k\| \\ &\leq \|x^k - z\|^2 + \underbrace{D^2 \|Rx^k\|^2 + 2D \nu(z) \|Rx^k\|}_{\varepsilon'_k(z)} \quad \forall k \in K_0 \cup K_1, z \in \mathbf{fix} T \end{aligned} \quad (3.12)$$

with $(\varepsilon'_k(z))_{k \in K_0 \cup K_1} \in \ell_+^1$ for all $z \in \mathbf{fix} T$. Combining this with (3.3) and telescoping the inequalities, we obtain that for all $z \in \mathbf{fix} T$

$$\|x^0 - z\|^2 \geq \sigma \sum_{k \in K_2} \|Rx^k\|^2 - \sum_{k \in K_0 \cup K_1} \varepsilon'_k(z). \quad (3.13)$$

Summability of $(\varepsilon'_k(z))_{k \in K_0 \cup K_1}$ implies that of $(\|Rx^k\|^2)_{k \in K_2}$; combined with the summability of $(\|Rx^k\|)_{k \in K_1 \cup K_2}$, it follows that the whole sequence of residuals is square-summable.

♠ **3.2(iii)**: follows combining **3.2(ii)** with **Thm. 2.7**.

♠ **3.2(iv)**: trivially follows from the already proven point **3.2(ii)**, together with the observation that since η_k is kept constant whenever $k \notin K_0$, the condition $\|Rx^k\| \leq c\eta_k$ will be satisfied infinitely often if $c > 0$.

♠ **3.2(v)**: suppose that $x^k \rightarrow x^*$ and that R is metrically subregular at x^* with modulus κ . Then, for some $\bar{k} \in \mathbb{N}$ it holds that $x^k \in B(x^*, \varepsilon)$ for all $k \geq \bar{k}$. Letting $z_k = \Pi_{\mathbf{fix} T} x^k$, clearly $z^k \rightarrow x^*$. Metric subregularity then reads

$$\|x^k - z^k\| = \mathbf{dist}(x^k, \mathbf{fix} T) = \mathbf{dist}(x^k, \mathbf{zer} R) \leq \kappa \|Rx^k\| \quad \forall k \geq \bar{k}. \quad (3.14)$$

Since R is 2α -Lipschitz continuous and $Rz^k = 0$ we also have

$$\|x^k - z^k\| \leq \kappa \|Rx^k\| \leq 2\alpha\kappa \|x^k - z^k\| \quad \forall k \geq \bar{k} \quad (3.15)$$

which means that $\mathbf{dist}(x^k, \mathbf{fix} T)$ and $\|Rx^k\|$ converge to 0 with the same local rate of convergence. Combining (3.3) and (3.14) we obtain

$$\|x^{k+1} - z^{k+1}\|^2 \leq \|x^{k+1} - z^k\|^2 \leq \|x^k - z^k\|^2 - \sigma \|Rx^k\|^2 \leq \rho^2 \|x^k - z^k\|^2 \quad \forall k \in K_2 \quad (3.16)$$

where $\rho := \sqrt{1 - \sigma/\kappa^2} \in (0, 1)$. Since the sequence $(\|Rx^k\|)_{k \in K_0 \cup K_1}$ is summable, it remains to show that so is $(\|Rx^k\|)_{k \in K_2}$: using (3.6) (3.15) and (3.16) we obtain

$$\begin{aligned} \frac{1}{2\alpha} \sum_{K_2 \ni k \geq \bar{k}} \|Rx^k\| &\leq \sum_{K_2 \ni k \geq \bar{k}} \|x^k - z^k\| \\ &\leq \kappa \sum_{K_2 \ni k \geq \bar{k}} \rho^{k - \kappa_2(k)} \|Rx^{\kappa_2(k)}\| \\ &\leq \kappa(1 + 2\alpha D) \sum_{K_2 \ni k \geq \bar{k}} \rho^{k - \kappa_2(k)} \|Rx^{\kappa_2(k)-1}\| \end{aligned}$$

and since $\kappa_2(k) - 1 \in K_0 \cup K_1$,

$$\leq \kappa(1 + 2\alpha D) \sum_{i \in \mathbb{N}} \rho^i \sum_{k \in K_0 \cup K_1} \|Rx^k\| < \infty \quad (3.17)$$

proving the claimed summability of the residuals. \square

3.1. Further generalizations. Though already quite broad and not restrictive, some hypothesis of Theorem 3.2 can be further relaxed without affecting the validity of the result. To avoid further complicating the framework we intentionally defer these minor details in the following side remarks.

Remark 3.3 (*Quasi-Fejér safeguard updates*). All claims in Thm.s 3.2(i) to 3.2(iv) remain valid if the Fejér condition (3.3) that characterizes safeguard updates at step 3(b) is relaxed to quasi-Fejér monotonicity. Namely, it can be replaced by

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \sigma \|Rx^k\|^2 + \varepsilon_k(z) \quad \forall z \in \mathbf{fix} T$$

where for all $z \in \mathbf{fix} T$ $(\varepsilon_k(z))_{k \in \mathbb{N}} \in \ell_+^1$ is a summable sequence. For Theorem 3.2(v) to hold too, it can be easily verified that it suffices to restrict $(\varepsilon_k(z))_{k \in \mathbb{N}} \in \ell_+^{1/2}$ for all $z \in \mathbf{fix} T$. \square

Remark 3.4 (No need for (3.4) for safeguard steps 3(b)). The fact that x^{k+1} is not too far from x^k ensures that the distance from $\mathbf{fix} T$ and the residual do not grow too much in between iterations (cf. (3.5) and (3.6)). However, these properties are only used for K_0 - and K_1 -updates, specifically in (3.11), (3.12) and (3.17) (in this last one observe that $\kappa_2(k) - 1 \notin K_2$). Therefore, Theorem 3.2 remains valid if (3.4) is only required for $k \in K_0 \cup K_1$, i.e., for iterates passing the test at either step 2 or 3(a). \square

Remark 3.5 (Less conservative r_{safe} for step 3(a)). As anticipated, the parameter r_{safe} is needed only to control the transition from K_2 - to K_1 -updates. This is evident from the proof of Theorem 3.2, where r_{safe} is only used in (3.10). To favor K_1 -updates we may therefore set $r_{\text{safe}} \leftarrow \infty$ after every K_0 -update at step 2 (there is no need to do so also after K_1 -updates). \square

Remark 3.6 (Omitting r_{safe} for monotone safeguard updates). In light of Remark 3.5, if the safeguard updates in step 3(b) are such that $\|Rx^{k+1}\| \leq \|Rx^k\|$, then the parameter r_{safe} plays no role and can be omitted. \square

3.2. Main idea. Being interested in solving the nonlinear equation (3.1), one could think of implementing one of the many existing fast methods for nonlinear equations that achieve fast asymptotic rates, such as Newton-type schemes. At each iteration, such schemes compute an update direction d^k and prescribe steps of the form $x^{k+1} = x^k + \tau_k d^k$, where $\tau_k > 0$ is a step-size that needs be sufficiently small

in order for the method to enjoy global convergence; on the other hand, fast asymptotic rates are ensured if $\tau_k = 1$ is eventually always accepted. The step-size is a crucial feature of fast methods, and a feasible τ_k is usually backtracked with a line-search over a smooth merit function. Unfortunately, in meaningful applications of the problem at hand arising from fixed-point theory the residual mapping R is nonsmooth, and the typical merit function $x \mapsto \|Rx\|^2$ does not meet the necessary smoothness requirement.

What we propose in this paper is a hybrid scheme that allows for the employment of any (fast) method for solving nonlinear equations, with global convergence guarantees that do not require smoothness, but which are based only on the nonexpansiveness of T . Computing directions $(d^k)_{k \in \mathbb{N}}$ with fast local methods for solving the nonlinear equation $Rx = 0$ in (3.1), Algorithm 1 can be specialized as follows:

- (1) *blind* updates as in step 2 shall be of the form $x^{k+1} = x^k + d^k$;
- (2) *educated* updates as in step 3(a) shall be of the form $x^{k+1} = x^k + \tau_k d^k$, with τ_k small enough so as to ensure the acceptance condition $\|Rx^{k+1}\| \leq \widehat{c} \|Rx^k\|$;
- (3) *safeguard* updates as in step 3(b) shall be employed as *last resort* both for globalization purposes and for well definedness of the scheme.

Ideally, the scheme should eventually reduce to the local scheme $x^{k+1} = x^k + d^k$ when *good* directions d^k are used.

In Section 4 we address the problem of providing explicit *safeguard* updates that comply with the quasi-Fejér monotonicity requirement of step 3(b). Because of the arbitrariness of the other two updates, once we succeed in this task Algorithm 1 will be of practical implementation. In Sections 5 and 6 we will then discuss specific K_0 - and K_1 -updates to be used at steps 2 and 3(a) that will make the scheme not only globally convergent, but also efficient and competitive with other state-of-the-art methods.

4. GENERALIZED MANN ITERATIONS

4.1. The classical Krasnosel'skiĭ-Mann scheme. Suppose that T and α are as in Assumption I. Starting from a point $x^0 \in \mathcal{H}$, the classical Krasnosel'skiĭ-Mann scheme (KM) performs the following updates

$$x^{k+1} = T_{\lambda_k} x^k = (1 - \lambda_k)x^k + \lambda_k T x^k \quad (4.1)$$

and weakly converges to a fixed point of T provided that $(\lambda_k)_{k \in \mathbb{N}} \subset [0, 1/\alpha]$, and that $(\lambda_k(1/\alpha - \lambda_k))_{k \in \mathbb{N}} \notin \ell^1$ [1, Thm. 5.14]. The key property of KM iterations is Fejér monotonicity: from Lemmas 2.3(iii), 2.3(vi) it can be easily inferred that x^{k+1} as in (4.1) satisfies

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \lambda_k(1/\alpha - \lambda_k)\|Rx^k\|^2 \quad \forall z \in \mathbf{fix} T.$$

In particular, in Algorithm 1 KM iterations can be used as *safeguard* updates at step 3(b) with $\varepsilon_k \equiv 0$. The drawback of such a selection is that it completely discards the hypothetical fast update direction d^k that *blind* and *educated* updates try to enforce. Though with suitable workarounds some fast asymptotic properties can theoretically still be proven, in practice classical KM updates significantly affect in a negative way the performance of Algorithm 1. This is particularly evident when the local method for computing the directions d^k is a *quasi-Newton* scheme; such methods are indeed very sensitive to past iterations, and discarding directions is neither theoretically sound nor beneficial in practice.

In this section we provide alternative *safeguard* updates that while ensuring the desirable Fejér monotonicity are also amenable to taking into account arbitrary directions. The effectiveness of this choice will both be proven in theory and backed

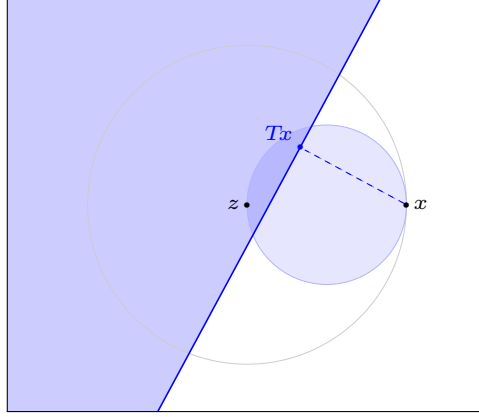


FIGURE 1. *Mann iteration of a FNE operator T as projection on C_x (the blue half-space, as defined in (4.2) for $\alpha = 1/2$). The outer circle is the set of all possible images of a nonexpansive operator, given that z is a fixed point. The inner circle corresponds to the possible images of *firmly* nonexpansive operators (cf. [9] for more details on this representation). Notice that C_x separates x from z as long as Tx is contained in the small circle, which characterizes firm nonexpansiveness.*

up by extensive numerical simulations later on in the paper. The key idea is based on a geometrical characterization of KM updates (4.1) which differs from the classical interpretation as (over-)relaxed fixed-point iterations of T .

Proposition 4.1 (KM iterations as projections). *Let T , R and α be as in Assumption I. For $x \in \mathcal{H}$, let*

$$C_x = C_x^{T,\alpha} := \{z \in \mathcal{H} \mid \|Rx\|^2 - 2\alpha\langle Rx, x - z \rangle \leq 0\}. \quad (4.2)$$

Then,

- (i) $x \in C_x$ iff $x \in \mathbf{fix} T$;
- (ii) $\mathbf{fix} T = \bigcap_{x \in \mathcal{H}} C_x$;
- (iii) for any $\lambda \in [0, 1/\alpha]$ it holds that $T_\lambda x = \Pi_{C_x, 2\alpha\lambda} x = (1 - 2\alpha\lambda)x + 2\alpha\lambda \Pi_{C_x} x$.

Proof. The set C_x can be equivalently expressed as

$$C_x = \{z \in \mathcal{H} \mid \langle x - T_{1/2\alpha} x, z - T_{1/2\alpha} x \rangle \leq 0\}$$

and 4.1(i) is of immediate verification. 4.1(ii) then follows from [1, Cor. 4.16] combined with Lem.s 2.2(d), 2.3(ii) and 2.3(iii).

We now show 4.1(iii). If $Rx = 0$, then $x \in \mathbf{fix} T$ and $C_x = \mathcal{H}$, and the claim is trivial. Otherwise, notice that $C_x = \{z \in \mathcal{H} \mid \langle z, Rx \rangle \leq \langle Rx, x - \frac{1}{2\alpha} Rx \rangle\}$ and the claim can be readily verified using the generic formula for the projection on a halfspace $H_{v,\beta} := \{z \in \mathcal{H} \mid \langle v, z \rangle \leq \beta\}$ given by

$$\Pi_{H_{v,\beta}} x = x - \frac{[\langle v, x \rangle - \beta]_+}{\|v\|^2} v \quad (4.3)$$

defined for any $v \in \mathcal{H} \setminus \{0\}$ and $\beta \in \mathbb{R}$ (cf. [1, Ex. 28.16(iii)]). \square

Figure 1 helps better visualize this property in the more easily understandable case in which T is FNE, i.e., when $\alpha = 1/2$.

4.2. Generalized Mann updates. Though particularly attractive for its simplicity, cheapness and globality, the KM scheme (4.1) finds its main drawback in its convergence rate, being it Q -linear at best and highly sensitive to ill-conditioning of the problem. In response to these issues, Algorithm 1 allows for the integration of fast local methods still ensuring global convergence properties. The efficiency of the resulting scheme, which will be proven later on, is based on an ad hoc selection of *safeguard* updates for step 3(b) which is based on the following generalization of Proposition 4.1.

Proposition 4.2. *Let T , R and α be as in Assumption I. Suppose that $x, w \in \mathcal{H}$ and $\mu \in [0, 1/\alpha]$ are such that*

$$\rho := \|Rw\|^2 - 2\alpha\langle Rw, w - T_\mu x \rangle > 0. \quad (4.4)$$

Let $C_w = C_w^{T, \alpha}$ be as in (4.2), and for $\lambda \in [0, 1/\alpha]$ let $x^+ := \Pi_{C_w, 2\alpha\lambda} T_\mu x$. Then,

$$x^+ = T_\mu x - \lambda \frac{\rho}{\|Rw\|^2} Rw \quad (4.5)$$

and, for all $z \in \mathbf{fix} T$,

$$\|x^+ - z\|^2 \leq \|x - z\|^2 - \mu(1/\alpha - \mu)\|Rx\|^2 - \lambda(1/\alpha - \lambda) \frac{\rho^2}{\|Rw\|^2}. \quad (4.6)$$

Proof. The expression (4.5) easily follows from (4.3), since by condition (4.4) the positive part in the formula may be omitted. We now show (4.6). Using Lem.s 2.3(ii), 2.3(iii) and 2.3(vi) and the fact that $\|x - T_\mu x\| = \mu\|Rx\|$ we have that

$$\|T_\mu x - z\|^2 \leq \|x - z\|^2 - \mu(1/\alpha - \mu)\|Rx\|^2 \quad \forall z \in \mathbf{fix} T$$

(also for $\mu = 0$ in which case $T_\mu = \text{Id}$). If $\lambda = 0$ then (4.6) is trivial. Otherwise, since $\mathbf{fix} T \subseteq C_w$ (cf. Prop. 4.1), the claim follows from the identity $\Pi_C = \frac{1}{2\alpha\lambda}(\Pi_{C, 2\alpha\lambda} - (1 - 2\alpha\lambda)\text{Id})$ and (2.3). \square

Notice that condition (4.4) is equivalent to $T_\mu x \notin C_w$. Therefore, the statement can be simply rephrased as the fact that whenever for some $w \in \mathcal{H}$ the point $T_\mu x$ lies outside of the half-space C_w , which we know contains $\mathbf{fix} T$ (cf. Prop. 4.1), projecting onto it gets us closer to $\mathbf{fix} T$. It goes beyond this, however, not only in considering an additional (over-)relaxation parameter λ but most importantly in providing the term $\lambda(1/\alpha - \lambda) \frac{\rho^2}{\|Rw\|^2}$ that furnishes a lower bound on how much x^+ is closer to $\mathbf{fix} T$ with respect to x .

It is important to remark that it is possible to select $\mu = 0$, so that $T_\mu = \text{Id}$ and no extra evaluation of T is performed. However, practical evidence suggests that selecting $\mu > 0$ drastically improves the performance not only in terms of iterations, but also in terms of number of operations and execution time. Observe further that for $\mu = 0$ and $w = x$ we obtain the classical KM update $x^+ = T_\lambda x$.

To have an intuition on how we intend to exploit this result, suppose that $d \in \mathcal{H}$ is a *fast* update direction at $T_\mu x$, and that there exists a step-size $\tau > 0$ such that $w = T_\mu x + \tau d$ satisfies (4.4). Then, the point $x^+ = \Pi_{C_w, 2\alpha\lambda} T_\mu x$ takes into account the fast direction d and at the same time is closer to $\mathbf{fix} T$.

To rely on the same convergence properties of the classical KM scheme, however, we need the scalar ρ in (4.4) to be *sufficiently* positive. In the next result, the cornerstone of our method albeit its simplicity, we formally state this needed property and show that, for *any* $d \in \mathcal{H}$ it is always possible to find a step-size $\tau > 0$ such that $w = T_\mu x + \tau d$ complies with our requirements.

Theorem 4.3. *Let T , R and α be as in Assumption I. Let $x \in \mathcal{H}$, $\mu \in [0, 1/\alpha]$, $\sigma \in (0, 1)$ and $d \in \mathcal{H}$ be fixed. If $T_\mu x \notin \mathbf{fix} T$, then there exists a $\bar{\tau} \in (0, 1]$ such that*

for all $\tau \in (0, \bar{\tau}]$ the point $w = T_\mu x + \tau d$ satisfies

$$\|Rw\|^2 - 2\alpha \langle Rw, w - T_\mu x \rangle \geq \sigma \|RT_\mu x\| \|Rw\|. \quad (4.7)$$

Proof. Contrary to the claim, suppose that for all $\varepsilon \in (0, 1]$ there exists $\tau_\varepsilon \in (0, \varepsilon]$ such that $w_\varepsilon = T_\mu x + \tau_\varepsilon d$ satisfies

$$\|Rw_\varepsilon\|^2 - 2\alpha \langle Rw_\varepsilon, w_\varepsilon - T_\mu x \rangle < \sigma \|RT_\mu x\| \|Rw_\varepsilon\|$$

Since $T : \mathcal{H} \rightarrow \mathcal{H}$ is continuous, taking the limit as $\varepsilon \rightarrow 0^+$ (so that $w_\varepsilon \rightarrow T_\mu x$) we arrive at $\|RT_\mu x\|^2 \leq \sigma \|RT_\mu x\|^2$, a contradiction since $\sigma \in (0, 1)$ and $RT_\mu x \neq 0$. \square

We now formally define the generalized KM update.

Definition 4.4 (Generalized KM update). *A GENERALIZED KM UPDATE (GKM) AT x ALONG d FOR THE α -AVERAGED OPERATOR $T : \mathcal{H} \rightarrow \mathcal{H}$ IS*

$$x^+ = \Pi_{C_w, 2\alpha\lambda} T_\mu x = T_\mu x - \lambda \frac{\rho}{\|Rw\|^2} Rw$$

where $C_w = C_w^{T, \alpha}$ is as in (4.2), $\lambda \in [0, 1/\alpha]$, $\mu \in [0, 1/\alpha]$, $\sigma \in (0, 1)$ and $w = T_\mu x + \tau d$ with τ small enough such that

$$\rho := \|Rw\|^2 - 2\alpha \langle Rw, w - T_\mu x \rangle \geq \sigma \|Rw\| \|RT_\mu x\|$$

as in Theorem 4.3.

In the next section we provide some graphical support to help better visualize a generalized KM iteration. This will also give some intuition that will lead to an enhancement of Theorem 4.3.

4.3. A graphical interpretation. In the proof of Theorem 4.3 we only made use of the continuity of T ; we now improve the result exploiting at full the averagedness of T . Specifically, we will provide lower bounds on the step-size $\bar{\tau}$ with the help of some graphical intuition. For a brief wordy introduction we assume that T is FNE, so that $\alpha = 1/2$ and the α 's in the formulas cancel out; because of Lemma 2.3(iv) this simply amounts to considering $T_{1/2\alpha}$ in place of T , and therefore it causes no loss of generality. For simplicity we also consider the case $\mu = 0$, so that $T_\mu x = x$; the general case will be discussed in detailed shortly after.

Using the notation of Theorem 4.3, notice that those w such that the inequality $\rho = \langle Rw, x - Tw \rangle > 0$ holds are by definition those for which x does not belong to C_w . Since $w \notin C_w$ for all $w \notin \text{fix } T$, it follows that in order for this to happen it is necessary and sufficient that C_w does not separate w from x . We start observing that the set

$$B_{x,w} := \{\bar{w} \mid \langle w - \bar{w}, x - \bar{w} \rangle \leq 0\} \quad (4.8)$$

is a ball with x and w as antidiagonal points (\bar{w} represents any “possible” Tw). This is evident once we pass to polar coordinates centered in $c = \frac{x+w}{2}$, namely,

$$\begin{aligned} B_{x,w} &= \{c + \rho u \mid \|u\| = 1, \rho > 0, \langle \frac{w-x}{2} - \rho u, \frac{x-w}{2} - \rho u \rangle \leq 0\} \\ &= \{c + \rho u \mid \|u\| = 1, \rho > 0, \rho^2 \leq \frac{1}{4} \|x - w\|^2\} \\ &= \{\bar{w} \mid \|\bar{w} - c\| \leq \frac{1}{2} \|x - w\|\}. \end{aligned}$$

By translating $B_{x,w}$ by a vector $Tx - x$ we obtain a region in which Tw must lie as prescribed by firm nonexpansiveness of T (cf. Figures 1 and 2). It follows that if $\|x - w\|$ is smaller than $\frac{1}{2} \|x - Tx\|$ then the two balls have empty intersection, and consequently $x \notin C_w$ independently of what Tw is. Figure 2 helps better visualize these claims.

In order for the line-search condition (4.7) to be satisfied, however, the condition $x \notin C_w$ is not enough, as x needs to be sufficiently far from C_w , where sufficiently is

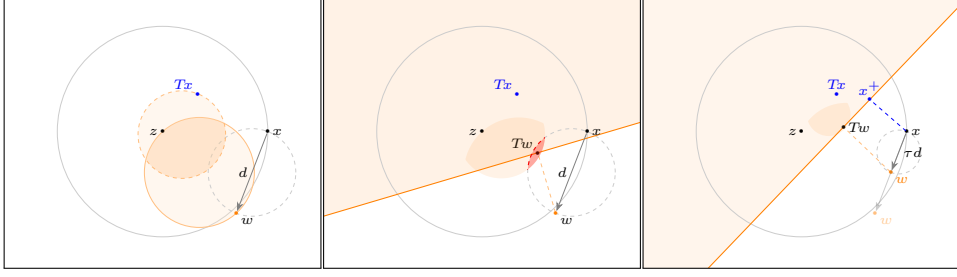


FIGURE 2. *SuperMann iteration of a FNE operator T as projection on C_w .* On the left, the darker orange region represents the area in which Tw must lie given the points x , Tx and the fixed point z as prescribed by firm nonexpansiveness of T . The central picture shows that if Tw lies (also) in the ball $B_{x,w}$ as in (4.8), then the half-space C_w (shaded in orange) separates x from w , which is to be avoided. In the rightmost figure, when w is close enough to x the feasible region for Tw has empty intersection with $B_{x,w}$ and indeed w and x are both outside of C_w .

somehow proportional to $\|Rw\|\|RT_\mu x\|$. The next result provides a generalization of this intuition extended to the case $\mu \neq 0$ and $\alpha \neq 1/2$.

Lemma 4.5. *Let T , R and α be as in [Assumption I](#). Consider an arbitrary point $y \in \mathcal{H}$ and $\sigma \in (0, 1)$. Then,*

$$\rho := \|Rw\|^2 - 2\alpha \langle Rw, w - y \rangle \geq \sigma \|Rw\| \|Ry\|$$

for all $w \in \mathcal{H}$ such that

$$\|w - y\| \leq \frac{1-\sigma}{4\alpha} \|Ry\|.$$

Proof. Let $w \in \mathcal{H}$ and a constant $c \geq 0$ to be determined be such that

$$\|w - y\| \leq c \|Ry\|.$$

Observe that $\rho = 4\alpha^2 \langle w - T_{1/2\alpha} w, y - T_{1/2\alpha} w \rangle$, and recall from [Lem.s 2.3\(ii\), 2.3\(iv\)](#) that $\text{Id} - T_{1/2\alpha} = \frac{1}{2\alpha} R$ and that $T_{1/2\alpha}$ is FNE. Then,

$$\rho = 4\alpha^2 (\|w - T_{1/2\alpha} w\|^2 + \langle w - T_{1/2\alpha} w, y - w \rangle)$$

using Cauchy-Schwartz inequality,

$$\geq 4\alpha^2 \|w - T_{1/2\alpha} w\| (\|w - T_{1/2\alpha} w\| - \|y - w\|)$$

the hypothesis on $\|y - w\|$,

$$\geq 2\alpha \|Rw\| (\|w - T_{1/2\alpha} w\| - \frac{1-\sigma}{2} \|y - T_{1/2\alpha} y\|)$$

the reverse triangular inequality,

$$\geq 2\alpha \|Rw\| (\frac{1+\sigma}{2} \|y - T_{1/2\alpha} y\| - \|(\text{Id} - T_{1/2\alpha})w - (\text{Id} - T_{1/2\alpha})y\|)$$

the nonexpansiveness of $\text{Id} - T_{1/2\alpha}$ (cf. [Lem. 2.2\(b\)](#)),

$$\geq 2\alpha \|Rw\| (\frac{1+\sigma}{2} \|y - T_{1/2\alpha} y\| - \|w - y\|)$$

and again the hypothesis on $\|w - x\|$,

$$\geq (1 - 4\alpha c) \|Rw\| \|Ry\|$$

we obtain that $\rho \geq \sigma \|Rw\| \|Ry\|$ as long as $\|w - y\| \leq \frac{1-\sigma}{4\alpha} \|Ry\|$. \square

Notice that for $\alpha = 1/2$ (i.e., for T FNE) and $\sigma = 0$ the inequality in [Lem. 4.5](#) reduces to $\langle w - Tw, x - Tw \rangle \geq 0$ as expected.

4.4. Convergence of GKM. We now show how iterating GKM steps gives rise to an algorithm that extends the KM scheme and maintains the same convergence properties.

In the steps we are about to state in the following result, we rule out the non interesting case in which $\|Rx^k\| = 0$ or $\|Rw^k\| = 0$ for some k , which corresponds to having found a solution to the problem (3.1) in a finite number of steps. Moreover, notice that setting $d^k \equiv 0$ yields the classical KM scheme.

Theorem 4.6 (GKM scheme). *Let T , R and α be as in Assumption 1. Let $\beta, \sigma \in (0, 1)$, $x^0 \in \mathcal{H}$, $(\lambda_k)_{k \in \mathbb{N}}, (\mu_k)_{k \in \mathbb{N}} \subset [0, 1/\alpha]$, and $(d^k)_{k \in \mathbb{N}} \subset \mathcal{H}$ be fixed. Starting from $k = 0$, consider the iterates generated by the generalized KM scheme (GKM)*

- 1: $w^k = T_{\mu_k} x^k + \tau_k d^k$
 where $\tau_k = \beta^{i_k}$, and $i_k \in \mathbb{N}$ is the smallest such that
 $\rho_k := \|Rw^k\|^2 - 2\alpha \langle Rw^k, w^k - T_{\mu_k} x^k \rangle \geq \sigma \|Rw^k\| \|RT_{\mu_k} x^k\|$
- 2: $x^{k+1} = T_{\mu_k} x^k - \lambda_k \frac{\rho_k}{\|Rw^k\|^2} Rw^k$
- 3: set $k \leftarrow k + 1$ and go to [step 1](#).

Then,

(i) $(x^k)_{k \in \mathbb{N}}$ is Fejér monotone with respect to $\mathbf{fix} T$, with

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \left\{ \mu_k(1/\alpha - \mu_k) + \sigma^2[1 - 2\alpha\mu_k]_+ \lambda_k(1/\alpha - \lambda_k) \right\} \|Rx^k\|^2.$$

for all $z \in \mathbf{fix} T$;

(ii) for all $k \in \mathbb{N}$, $\tau_k = 1$ if $d^k = 0$, and $\tau_k \geq \min \left\{ \beta \frac{1-\sigma}{4\alpha\|d^k\|}, 1 \right\}$ otherwise.

Moreover, if for some $\delta > 0$ it holds that for all k either $\delta \leq \mu_k \leq 1/\alpha - \delta$, or both $\delta \leq \lambda_k \leq 1/\alpha - \delta$ and $\mu_k \leq 1/2\alpha - \delta$, then the following also hold:

(iii) $Rx^k \rightarrow 0$ with $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^2$;

(iv) $(x^k)_{k \in \mathbb{N}}$ converges weakly to a point $x^* \in \mathbf{fix} T$.

If, additionally $(x^k)_{k \in \mathbb{N}}$ is strongly convergent (to x^*) and R is metric subregular at x^* , then

(v) $\mathbf{dist}(x^k, \mathbf{fix} T) \rightarrow 0$ Q -linearly and $\|Rx^k\| \rightarrow 0$ R -linearly.

Proof.

♠ **4.6(i)**: from (4.6) and the lower bound on ρ_k at [step 1](#) we obtain

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \mu_k(1/\alpha - \mu_k) \|Rx^k\|^2 - \sigma^2 \lambda_k(1/\alpha - \lambda_k) \|RT_{\mu_k} x^k\|^2.$$

Since R is 2α -Lipschitz continuous, we also have that

$$\|Rx - RT_{\mu_k} x\| \leq 2\alpha \|x - T_{\mu_k} x\| = 2\alpha \mu_k \|Rx\|,$$

hence $\|RT_{\mu_k} x\| \geq [1 - 2\alpha\mu_k]_+ \|Rx\|$, and the proof follows combining the two inequalities.

♠ **4.6(ii)**: follows from [Lem. 4.5](#) together with the minimality of $i_k \in \mathbb{N}$ in determining the step-size $\tau_k = \beta^{i_k}$; in particular, a feasible τ_k is always obtained in finitely many backtrackings, proving the algorithm to be well defined.

♠ **4.6(iii)** and **4.6(iv)**: under the assumptions on $(\lambda_k)_{k \in \mathbb{N}}$ and $(\mu_k)_{k \in \mathbb{N}}$, from **4.6(i)** it follows that $\sigma_k \geq \underline{\sigma}$ for some $\underline{\sigma} > 0$. In particular, the algorithm is a special instance of [Alg. 1](#) with $\sigma = \underline{\sigma}$ and $\varepsilon_k \equiv c = \widehat{c} = 0$ for all k . The sought proof then follows from [Thm. 3.2\(ii\)](#) and [Thm. 3.2\(iii\)](#), respectively, combined with [Rem. 3.4](#).

♠ **4.6(v)**: suppose now that convergence is strong and that R is metrically subregular at the limit x^* with modulus κ and radius ε . Let $\bar{k} \in \mathbb{N}$ be such that

$x^{\bar{k}} \in B(x^*, \varepsilon)$ for all $k \geq \bar{k}$, and let $z^k := \Pi_{\mathbf{fix} T} x^k$, whose well-definedness is due to [Lem. 2.3\(i\)](#). Metric subregularity then reads

$$\|x^k - z^k\| = \mathbf{dist}(x^k, \mathbf{fix} T) = \mathbf{dist}(x^k, \mathbf{zer} R) \leq \kappa \|Rx^k\| \quad \forall k \geq \bar{k}.$$

From [4.6\(i\)](#) we obtain that for all $k \geq \bar{k}$

$$\begin{aligned} \|x^{k+1} - z^{k+1}\|^2 &\leq \|x^{k+1} - z^k\|^2 \leq \|x^k - z^k\|^2 - \underline{\sigma} \|Rx^k\|^2 \\ &\leq \|x^k - z^k\|^2 - \frac{\underline{\sigma}}{\kappa^2} \|x^k - z^k\|^2 \end{aligned}$$

and therefore

$$\mathbf{dist}(x^{k+1}, \mathbf{fix} T) \leq \sqrt{1 - \frac{\underline{\sigma}}{\kappa^2}} \mathbf{dist}(x^k, \mathbf{fix} T)$$

proving Q -linear convergence rate of $(\mathbf{dist}(x^k, \mathbf{fix} T))_{k \in \mathbb{N}}$. R -linear convergence of $(\|Rx^k\|)_{k \in \mathbb{N}}$ follows from the bound $\|Rx^k\| \leq 2\alpha \|x^k - z^k\|$ due to 2α -Lipschitz continuity of R . \square

4.5. Comparison with Solodov & Svaiter's. GKM iterations are in the same flavor of the updates in [\[25, Alg. 2.1\]](#). The cited scheme solves [\(3.1\)](#) where R is a continuous monotone operator, not necessarily the residual of a nonexpansive one. [Algorithm 2](#) translates it into our framework, where R is instead the residual of a FNE operator T . The parameter $\sigma_k > 0$ depends on the positive definite matrix B_k in such a way that the line-search [\(4.9\)](#) is feasible.

Algorithm 2 Solodov & Svaiter's [\[25, Alg. 2.1\]](#). (*Exact Globalized Newton Method for Monotone Equations*)

- 1: Select a pos.def. matrix B_k and let $d^k = -B_k^{-1} Rx^k$
- 2: $w^k = x^k + \tau_k d^k$ with $\tau_k \in (0, 1]$ s.t. $-\langle Rw^k, d^k \rangle \geq \sigma_k \|d^k\|^2$, i.e.,

$$\|Rw^k\|^2 - \langle Rw^k, x^k - Tw^k \rangle \leq -\sigma_k \tau_k \|d^k\|^2 \quad (4.9)$$

- 3: $x^{k+1} = \Pi_{H_k} x^k$ where

$$\begin{aligned} H_k &:= \{z \in \mathcal{H} \mid \langle Rw^k, z - w^k \rangle \leq 0\} \\ &= \{z \in \mathcal{H} \mid \|Rw^k\|^2 - \langle Rw^k, z - Tw^k \rangle \geq 0\} \end{aligned} \quad (4.10)$$

- 4: $k \leftarrow k + 1$, and go to [step 1](#)
-

Remark 4.7 (Half-spaces in S&S and GKM). Denoting $C_k = C_{w^k}$ as in [Prop. 4.2](#) so that $x^{k+1} = \Pi_{C_k} x^k$ in the GKM scheme (where for simplicity we choose $\lambda_k = 1$ and $\mu_k = 0$), for the half-spaces [\(4.10\)](#) it holds that

$$\mathbf{zer} R \subseteq C_k \subseteq H_k,$$

the last inclusion holding as equality iff $Rw^k = 0$. This means that in the GKM scheme, the same w^k yields an iterate x^{k+1} which is closer to $\mathbf{zer} R$ with respect to S&S's update (cf. [Figure 3](#)). Notice further that the hyperplanes delimiting the two half-spaces are parallel, with ∂C_k passing by Tw^k and ∂H_k by w^k . \square

Remark 4.8 (Directions in S&S and GKM). Solodov & Svaiter impose specific choices for the direction d^k (cf. [Figure 3](#)) which is due to the fact that they address a broader class of operators for which the geometrical properties that led to the GKM scheme do not apply. Because of our specialization to FNE operators (recall that we always reduce to the case of FNE operators through the suitable $1/2\alpha$ -averaging), we instead have complete freedom in selecting d^k . [Algorithm 2](#) in its extended version allows for a range of inexactness in computing d^k , however our scheme is robust to *any* error. \square

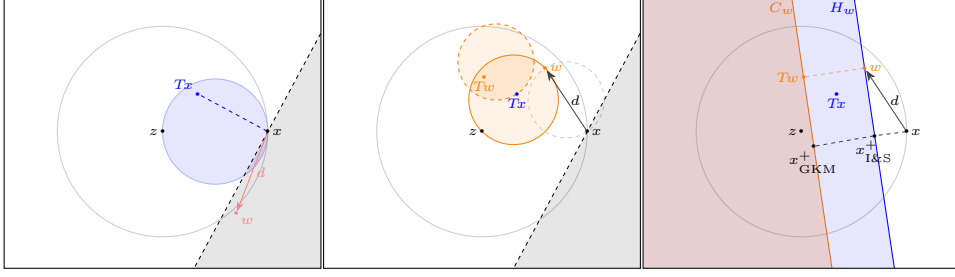


FIGURE 3. The *descent condition* (4.9) prevents the update directions d^k in S&S's scheme to point in the gray-shaded area. As a result, differently from the GKM scheme, S&S is not robust to any choice of direction and cannot accept for instance the (bad) direction as in Figure 2. In any case, the half-space C_w onto which x is projected according to the GKM scheme is properly contained in the half-space H_w corresponding to the S&S's update. As a result, the GKM update is always closer than S&S's to the set of solutions.

Algorithm 3 *SuperMann* scheme for solving (3.1), given an α -averaged operator T with residual $R = \text{Id} - T$

REQUIRE $x^0 \in \mathcal{H}$, $c, \hat{c} \in [0, 1)$, $\beta, \sigma \in (0, 1)$, $\lambda \in (0, 1/\alpha)$, $\mu \in [0, 1/\alpha)$.

INITIALIZE $\eta_0 = r_{\text{safe}} = \|Rx^0\|$, $k = 0$

1. IF $Rx^k = 0$, THEN STOP.
 2. Choose an update direction $d^k \in \mathcal{H}$
 3. (**K₀**) IF $\|Rx^k\| \leq c\eta_k$, THEN set $\eta_{k+1} = \|Rx^k\|$, $r_{\text{safe}} = \infty$, proceed with a *blind update* $x^{k+1} = w^k := x^k + d^k$ and go to step 6.
 4. Set $\eta_{k+1} = \eta_k$ and $\tau_k = 1$
 5. Let $w^k = T_\mu x^k + \tau_k d^k$.
 - 5(a) (**K₁**) IF the *safe condition* $\|Rx^k\| \leq r_{\text{safe}}$ holds and w^k is *educated*:

$$\|Rw^k\| \leq \hat{c} \|Rx^k\|$$
 THEN set $x^{k+1} = w^k$ and go to step 6.
 - 5(b) (**K₂**) IF $\rho_k := \|Rw^k\|^2 - 2\alpha \langle Rw^k, w^k - x^k \rangle \geq \sigma \|Rw^k\| \|RT_\mu x^k\|$ THEN set

$$x^{k+1} = T_\mu x^k - \lambda \frac{\rho_k}{\|Rw^k\|^2} Rw^k$$
 OTHERWISE set $\tau_k \leftarrow \beta \tau_k$ and go back to step 5.
 IF x^k was not computed with another *GKM* update 5(b), THEN update $r_{\text{safe}} = \|Rx^k\| + \hat{\varepsilon}_k$ for some $\hat{\varepsilon}_k \geq 0$.
 6. Set $k \leftarrow k + 1$ and go to step 1.
-

5. THE SUPERMANN SCHEME

We now address the problem of characterizing quality *blind* and *educated* update directions; in Section 6 we will then provide specific examples. Integrating GKM updates as in Definition 4.4 into Algorithm 1 gives rise to the *SuperMann scheme* (Alg. 3). To discuss its global and local convergence properties we stick to the same notation of the general framework of Algorithm 1, denoting the sets of *blind*, *educated*, and *safeguard* updates as K_0 , K_1 and K_2 , respectively.

5.1. Global and local convergence. To comply with (3.4), we impose the following upper bound on the selected directions.

Assumption II. *There exists a constant $D \geq 0$ such that the directions $(d^k)_{k \in \mathbb{N}}$ in the *SuperMann scheme* (Alg. 3) satisfy*

$$\|d^k\| \leq D\|Rx^k\| \quad \forall k \in \mathbb{N}. \quad (5.1)$$

Theorem 5.1 (Global convergence of the *SuperMann scheme*). *Let T , R and α be as in Assumption I. Consider the iterates generated by the *SuperMann scheme* (Alg. 3) with $(d^k)_{k \in \mathbb{N}}$ selected so as to satisfy Assumption II. Then,*

- (i) $(x^k)_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to $\mathbf{fix} T$;
- (ii) for all $k \in \mathbb{N}$, $\tau_k = 1$ if $d^k = 0$, and $\tau_k \geq \min \left\{ \beta \frac{1-\sigma}{4\alpha\|d^k\|}, 1 \right\}$ otherwise.
- (iii) $Rx^k \rightarrow 0$ with $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^2$;
- (iv) $(x^k)_{k \in \mathbb{N}}$ converges weakly to a point $x^* \in \mathbf{fix} T$.
- (v) if $c > 0$, then the number of blind updates at step 3 is infinite.

If, additionally, $x^k \rightarrow x^*$ (this being true if \mathcal{H} is finite-dimensional), and R is metric subregular at x^* , then

- (vi) $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^1$.

Proof. Because of Thm. 4.3 we know that for any arbitrary direction d^k a feasible step-size τ_k complying (at least) with the requirements of step 5(b) will eventually be found, lower bounded as in 5.1(ii) due to Thm. 4.6(ii). In particular, the scheme is well defined. Moreover, from Thm. 4.6(i) we have that there exists a constant $\underline{\sigma} > 0$ such that

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \underline{\sigma}\|Rx^k\|^2 \quad \text{for all } k \in K_2 \text{ and } z \in \mathbf{fix} T.$$

It follows that the *SuperMann scheme* is a special case of Alg. 1 and the proof entirely follows from Thm. 3.2 and Rem.s 3.4, 3.5. \square

The complexity of the classical KM scheme (4.1) simply amounts to evaluations of T , while that of the *SuperMann scheme* also depends on the oracle for computing the update directions d^k . As it was proven in Theorem 5.1, for the sake of global convergence $(d^k)_{k \in \mathbb{N}}$ can pretty much be anything; however, in the next result we show that more sophisticated choices make the scheme much more appealing for asymptotic properties. Being interested in maintaining the computational simplicity of the classical KM scheme we focus on quasi-Newton directions since their computation only requires linear algebra. For this reason, in the following Theorem 5.2 we will provide a condition for superlinear convergence which is equivalent to the famous result by Dennis and Moré, which in turn will be used in Section 6 for deriving a tailored quasi-Newton scheme based on the famous quasi-Newton Broyden's method.

Theorem 5.2 (Local convergence of the *SuperMann scheme*). *Let T , R and α be as in Assumption I. Consider the iterates generated by the *SuperMann scheme* (Alg. 3) with $\hat{c} \in (0, 1)$, and suppose that $(x^k)_{k \in \mathbb{N}}$ converges strongly to a point x^* at which R is strictly differentiable. Suppose further that the update directions $(d^k)_{k \in \mathbb{N}}$ satisfy (5.1) and the Dennis-Moré condition*

$$\lim_{k \rightarrow \infty} \frac{\|RT_\mu x^k + JR(x^*)d^k\|}{\|d^k\|} = 0. \quad (5.2)$$

Then,

- (i) eventually, step-size $\tau_k = 1$ will be always accepted and safeguard updates K_2 never performed (i.e., the scheme reduces to the local method $x^{k+1} = x^k + d^k$);

- (ii) $Rx^k \rightarrow 0$ at Q -superlinear rate;
 - (iii) if $c > 0$, then the complement of K_0 is finite.
- Moreover, if $JR(x^*)$ is strongly nonsingular, then
- (iv) $x^k \rightarrow 0$ at R -superlinear rate.

Proof. Let $w_0^k = T_\mu x^k + d^k$. Since $x^k \rightarrow x^*$ and $\|Rx^k\| \rightarrow 0$ by [Thm. 5.1\(iii\)](#), then $T_\mu x^k \rightarrow x^*$ and

$$\|w_0^k - x^*\| \leq \|T_\mu x^k - x^*\| + \|d^k\| \stackrel{(5.1)}{\leq} \|T_\mu x^k - x^*\| + D\|Rx^k\| \rightarrow 0.$$

Now,

$$0 \leftarrow \frac{RT_\mu x^k + JR(x^*)d^k}{\|d^k\|} = \frac{RT_\mu x^k + JR(x^*)(T_\mu x^k - w_0^k) - Rw_0^k}{\|w_0^k - T_\mu x^k\|} + \frac{Rw_0^k}{\|d^k\|}$$

The first term goes to zero because of strict differentiability of R at x^* and the fact that $(T_\mu x^k, w_0^k) \rightarrow (x^*, x^*)$. Therefore,

$$0 \leftarrow \frac{\|Rw_0^k\|}{\|d^k\|} \stackrel{(5.1)}{\geq} \frac{\|Rw_0^k\|}{D\|Rx^k\|}$$

In particular, there exists $\bar{k} \in \mathbb{N}$ such that $\|Rw_0^k\| \leq \hat{c}\|Rx^k\|$ for all $k \geq \bar{k}$. Therefore, for such k the point $w_0^k = T_\mu x^k + d^k$ will always pass the test for entering K_1 , resulting in $x^{k+1} = w_0^k = T_\mu x^k + d^k$ for all $k \geq \bar{k}$; this shows [5.2\(i\)](#). Moreover, the limit above can be reformulated as

$$\lim_{k \rightarrow \infty} \frac{\|Rx^{k+1}\|}{\|Rx^k\|} = 0$$

proving [5.2\(ii\)](#).

If $c > 0$, by possibly enlarging \bar{k} we have that $\|Rx^k\| \leq c\|Rx^{k-1}\|$ for all $k \geq \bar{k}$. Let $k_\ell \in K_0$ be such that $k_\ell \geq \bar{k}$, which exists as ensured by [Thm. 5.1\(v\)](#); then, $\|Rx^{k_\ell+1}\| \leq c\|Rx^{k_\ell}\| = c\eta_{k_\ell+1}$ (the last equality being due to the fact that $k_\ell \in K_0$) proving $k_\ell + 1 \in K_0$ as well, hence [5.2\(iii\)](#).

If G_\star is strongly nonsingular, then by [Lem. 2.8](#) and since $x^k \rightarrow x^*$, there exists $\kappa > 0$ such that eventually $\|x^k - x^*\| \leq \frac{1}{\kappa}\|Rx^k\|$, and [5.2\(iv\)](#) follows from the already proven point [5.2\(ii\)](#). \square

In [Theorem 5.2\(i\)](#) we proved that when $(d^k)_{k \in \mathbb{N}}$ are *good* directions, the *SuperMann scheme* eventually reduces to the (fast) local method $x^{k+1} = x^k + d^k$. As anticipated we are mostly concerned with quasi-Newton directions; nevertheless, we the freedom in the selection of $(d^k)_{k \in \mathbb{N}}$ allows for full elasticity in trading-off efficiency of the update directions and computational complexity. Without going into detail, this means, for instance, that should (generalized) first-order information of the residual R be easily computable, semismooth Newton methods could be chosen for computing d^k and asymptotic quadratic convergence under standard assumptions at the solution be achieved.

5.2. Comparisons with other methods.

5.2.1. Hybrid global and local phase algorithms. *Blind* K_0 -updates in *SuperMann scheme* are inspired from [\[2, Alg. 1\]](#), and so is the notation $K_0 = \{k_0, k_1, \dots\}$.

Educated K_1 - and *safeguard* K_2 -updates instead play the role of *inner*- and *outer*-phases in the general algorithmic framework described in [\[15, §5.3\]](#) for finding a zero of a candidate merit function φ (e.g. $\varphi(x) = \frac{1}{2}\|Rx\|^2$ in our case). Differently from [\[15, Alg. 5.16\]](#) where all previous inner-phase iterations are discarded as soon as the required sufficient decrease is not met, the *SuperMann scheme* allows for an alternation of phases that eventually stabilizes on the fast local one, provided the solution is sufficiently regular. Our scheme is more in the flavor of [\[15, Alg.](#)

5.19], although with less conservative requirements for triggering *inner* K_1 -updates ($\varphi(x^{k+1})$ is here compared with $\varphi(x^k)$, whereas in the cited scheme with the smallest past value).

5.2.2. *Line-search for KM.* The recent work [12] proposes an acceleration of the classical KM scheme for finding a fixed point of an α -averaged operator T based on a line-search on the relaxation parameter. Namely, instead of the *nominal* update $\bar{x} = T_\lambda x$ with $\lambda \in [0, 1/\alpha]$ as in (4.1), values $\lambda' > 1/\alpha$ are first tested and the update $x^+ = T_{\lambda'} x$ is accepted as long as $\|Rx^+\| \leq \hat{c}\|R\bar{x}\|$ holds for some constant $\hat{c} \in (0, 1)$. Convergence is significantly enhanced in practice for many applications, and the method is particularly attractive when $T = S_2 \circ S_1$ is the composition of an affine mapping S_1 and a cheap operator S_2 , in which case the line-search is computationally inexpensive.

In the setting of the *SuperMann scheme*, this corresponds to selecting $d^k = -Rx^k$, discarding *blind* updates (*i.e.*, setting $c = 0$), foretracking *educated* updates and using plain KM iterations as safeguard steps. However, though preserving the same theoretical convergence guarantees of KM (hence of the *SuperMann scheme*), it does not improve its best-case local linear rate. Moreover, *educated* updates as in [12] compare with the *nominal* KM safeguard, resulting in a possibly more conservative line-search — since $\|R(Tx)\| \leq \|Rx\|$ (cf. Lem. 2.3(v)), and in a waste of KM iterations whenever nominal steps are discarded.

5.2.3. *Smooth optimization with envelope functions.* For solving nonsmooth minimization problems in composite form, [20, 21] introduced *forward-backward envelope* (FBE) and *Douglas-Rachford envelope* (DRE) functions. The original nonsmooth problem is recast into the minimization of continuous (possibly continuously differentiable) real-valued exact penalty functions closely related to forward-backward and Douglas-Rachford splittings, named *envelopes* due to their kinship with the Moreau *envelope* and the Proximal Point Algorithm. This paved the way for the employment of fast methods for smooth unconstrained minimization problems [20, 21, 26], or for globalizing convergence of fast methods for solving nonlinear equations [27].

Though envelope functions have proven very effective, their employment is limited to composite operators as described above. The *SuperMann scheme* instead offers a unifying framework that is based uniquely on evaluations of the nonexpansive mapping T , regardless of their structure.

6. A MODIFIED BROYDEN'S SCHEME

For efficiently solving a nonlinear equation

$$\text{given } F : \mathcal{H} \rightarrow \mathcal{H} \quad \text{find } x^* \in \mathbf{zer} F, \quad (6.1)$$

starting from an invertible operator $B_0 \in \mathcal{B}(\mathcal{H})$ quasi-Newton methods operate low rank updates satisfying the *secant condition*

$$B_{k+1}s_k = y_k \quad \text{with} \quad \begin{cases} s_k = x^{k+1} - x^k \\ y_k = F(x^{k+1}) - F(x^k) \end{cases} \quad (6.2)$$

and prescribe recursive updates of the form

$$x^{k+1} = x^k - B_k^{-1}F(x^k). \quad (6.3)$$

Under some regularity condition of F at the solution, quasi-Newton schemes are locally superlinearly convergent yet, differently from Newton methods, without requiring the computation of first-order information. There are many quasi-Newton updates available, the BFGS formula being the most popular, but many of them

are provably well performing only under the assumption of symmetricity (and non-singularity) of the Jacobian at the solution. Symmetricity is ensured, for instance, when the objective is the minimization of a convex smooth function f ; the problem is indeed equivalent to solving the nonlinear equation $\nabla f(x) = 0$, whose Jacobian is a symmetric matrix provided $f \in C^2$.

Unfortunately this is often not the case, as it happens for meaningful applications of problem (3.1). Broyden's method offers a universal alternative that does not necessitate symmetricity; the standard Broyden's update is of the form

$$B_{k+1} = B_k + \frac{(y_k - B_k s_k)}{\|s_k\|^2} \otimes s_k \quad (6.4)$$

which is simply the operator *closest* to B_k that satisfies (6.2), namely,

$$B_{k+1} = \Pi_{S_k}(B_k) \quad \text{where} \quad S_k := \{B \in \mathcal{B}(\mathcal{H}) \mid B s_k = y_k\} \quad (6.5)$$

The scheme is well-defined as long as the scalar products $\langle B_k^{-1} y_k, s_k \rangle$ are not null.

Lemma 6.1. *Suppose $B_k \in \mathcal{B}(\mathcal{H})$ is invertible, and let B_{k+1} be given by the Broyden's update (6.4). If $\langle B_k^{-1} y_k, s_k \rangle \neq 0$, then B_{k+1} is invertible.*

Proof. If $\langle B_k^{-1} y_k, s_k \rangle \neq 0$, then it is well defined the operator

$$B_k^{-1} + \frac{s_k - B_k^{-1} y_k}{\langle B_k^{-1} y_k, s_k \rangle} \otimes ((B_k^{-1})^* s_k) \quad (6.6)$$

which can be readily verified to be the inverse of B_{k+1} . \square

With a suitable modification, first introduced in [22] for the solution of linear systems and later generalized in [16] also for locally smooth equations, it is possible to enforce the needed properties while preserving the local superlinear convergence of Broyden's scheme. [24] extended the result to infinite-dimensional systems of locally smooth equations, but requiring the first Broyden's estimate B_0 to be *sufficiently close* to the Jacobian at the solution G_* . More precisely, $B_0 - G_*$ needs be a Hilbert-Schmidt operator or, in other words, of finite Frobenius norm (a property that trivially holds in finite dimension).

In this section we show that the modified scheme still enjoys the same properties for infinite-dimensional systems of locally semidifferentiable equations without any assumption on B_0 other than invertibility. Our proofs are based on the geometrical interpretation (6.5) and on the (firm) nonexpansiveness of the projection operator. We first consider the general framework (6.1) and then tailor the results for the employment of the modified Broyden's scheme in the *SuperMann scheme*.

6.1. Relaxation. We consider including a relaxation parameter in Broyden's update which shall be tuned in such a way to enforce some wanted properties on Broyden's operators. In Section 6.2 we will discuss specific such choices. In general, given $(\theta_k)_{k \in \mathbb{N}} \subset [0, 2]$ we modify Broyden's update as follows:

$$B_{k+1} = B_k + \theta_k \frac{(y_k - B_k s_k)}{\|s_k\|^2} \otimes s_k. \quad (6.7a)$$

Equivalently, letting $\tilde{y}_k = (1 - \theta_k) B_k s_k + \theta_k y_k$,

$$B_{k+1} = B_k + \frac{(\tilde{y}_k - B_k s_k)}{\|s_k\|^2} \otimes s_k \quad (6.7b)$$

resulting in the *relaxed* secant equation

$$B_{k+1} s_k = \tilde{y}_k. \quad (6.8)$$

Consequently, (6.5) becomes

$$B_{k+1} = \Pi_{\tilde{S}_k}(B_k) = \Pi_{S_k, \theta_k}(B_k) \quad \text{where} \quad \tilde{S}_k := \{B \in \mathcal{B}(\mathcal{H}) \mid B s_k = \tilde{y}_k\}. \quad (6.9)$$

From (6.8) the following modification of Lemma 6.1 immediately follows.

Lemma 6.2. *Suppose $B_k \in \mathcal{B}(\mathcal{H})$ is invertible, and let B_{k+1} be given by the relaxed Broyden's update (6.7). If $\langle B_k^{-1} \tilde{y}_k, s_k \rangle \neq 0$, then B_{k+1} is invertible.*

6.2. Enforcing properties. To enforce nonsingularity we proceed as in [22] and for a fixed parameter $\bar{\theta} \in (0, 1)$ we define

$$\gamma_k := \frac{\langle B_k^{-1} y_k, s_k \rangle}{\|s_k\|^2} \quad \text{and} \quad \theta_k := \begin{cases} 1 & \text{if } |\gamma_k| \geq \bar{\theta} \\ \frac{1 - \text{sgn}(\gamma_k) \bar{\theta}}{1 - \gamma_k} & \text{if } |\gamma_k| < \bar{\theta} \end{cases} \quad (6.10a)$$

with the convention $\text{sgn } 0 = 1$. If B_k is invertible, it can be readily verified that if $s_k \neq 0$, i.e., if $x^k \notin \text{fix } T$, then $\langle B_k^{-1} \tilde{y}_k, s_k \rangle \neq 0$, and therefore B_{k+1} is invertible too as ensured by Lemma 6.2. Moreover, notice that by definition of θ_k it holds that

$$1 - \bar{\theta} \leq \theta_k < 1 + \bar{\theta}. \quad (6.10b)$$

6.3. Superlinear convergence. From [6, Thm. 2.2] and [16, Thm. 2.1] it follows that the modified Broyden scheme (6.7) with $(\theta_k)_{k \in \mathbb{N}}$ selected as in (6.10a) satisfies the Dennis-Moré condition (5.2) provided that the mapping whose zeros are sought, R in our case, is continuously differentiable around the limiting point, be it x^* . In [14] the requirements for the Dennis-Moré condition to be satisfied for quasi-Newton schemes are relaxed to semidifferentiability around x^* and calmness of the semiderivative at x^* (cf. Section 2.4). All these results are proven only for $\mathcal{H} = \mathbb{R}^n$; we now extend their validity to arbitrary Hilbert spaces.

Theorem 6.3 (Broyden's method). *Suppose that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is calmly semidifferentiable at $x^* \in \text{zer } F$, and let $G_* = JF(x^*)$. Starting from $x^0 \in U_{x^*}$, let $(x^k)_{k \in \mathbb{N}}$ be the sequence recursively defined as in (6.3) with B_k given by the modified Broyden's update (6.7) with $(\theta_k)_{k \in \mathbb{N}} \subset [\bar{\theta}, 2 - \bar{\theta}]$ for some $\bar{\theta} > 0$. Suppose the operators B_k are all nonsingular, and that $(x^k)_{k \in \mathbb{N}}$ converges to x^* with $(\|x^k - x^*\|)_{k \in \mathbb{N}} \in \ell^1$.*

Then, either $F(x^k) = 0$ for some k , or the Dennis-Moré condition holds

$$\lim_{k \rightarrow \infty} \frac{\|(B_k - G_*)s_k\|}{\|s_k\|} = 0.$$

Proof. See Appendix A. □

Remark 6.4. It is evident from its proof that Theorem 6.3 remains valid if $(B_k)_{k \in \mathbb{N}}$ is updated using $s_k = w^k - x^k$ and $y_k = Fw^k - Fx^k$, where $(w^k)_{k \in \mathbb{N}}$ is an arbitrary sequence such that $(\|w^k - x^*\|)_{k \in \mathbb{N}} \in \ell^1$. To see this, simply trace the proof of Theorem 6.3 replacing any occurrence of x^k with w^k , yet leaving those of x^{k+1} unchanged. □

Theorem 6.5 (Superlinear convergence of the SuperMann scheme with relaxed Broyden's method). *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be an α -averaged operator for some $\alpha \in (0, 1]$, with $\text{fix } T \neq \emptyset$. Let $R := \text{Id} - T$ and let $(x^k)_{k \in \mathbb{N}}$ be the sequence generated by the SuperMann scheme (Alg. 3) with $d^k = B_k^{-1} R T_\mu x^k$ for all k , $(B_k)_{k \in \mathbb{N}}$ being chosen according to the modified Broyden's scheme of Section 6.2 with*

$$s_k = w^k - T_\mu x^k \quad \text{and} \quad y_k = R w^k - R T_\mu x^k$$

starting from an invertible operator $B_0 \in \mathcal{B}(\mathcal{H})$. Suppose that $(\|B_k^{-1}\|)_{k \in \mathbb{N}}$ remains bounded, and that $(x^k)_{k \in \mathbb{N}}$ converges strongly to a point x^ at which R is calmly semidifferentiable with $JR(x^*)$ strongly nonsingular.*

Then, $(d^k)_{k \in \mathbb{N}}$ satisfies (5.2), i.e., the directions are superlinearly convergent, and in particular Theorem 5.2 applies.

Proof. See Appendix A. □

7. SIMULATIONS

We conclude with some numerical examples to give tangible evidence of the robustifying and enhancing effect that the *SuperMann scheme* has on fixed-point iterations. In all simulations we deactivated blind updates by setting $c = 0$, and we selected $\sigma = 10^{-3}$ for safeguard updates and $\hat{c} = 1 - \sigma$ for educated updates. As for the summable sequence $\hat{\varepsilon}_k$ that determines how the safeguard parameter r_{safe} is updated we set $\hat{\varepsilon}_k = \min\{\|Rx^0\|, 1\}(1 - \sigma)^k$. Due to problem size we used limited-memory modified Broyden's directions with a memory buffer of 20 vectors, and as a consequence we cannot infer superlinear convergence from the theory, which instead is based on full-memory methods. This is a common issue also in classical large scale smooth unconstrained minimization, where however limited-memory methods such as L-BFGS are the ones of choice due to their outstanding performance in practice. This fact is equally evident in our simulations, where limited-memory Broyden directions yield an extremely “steep” linear convergence.

7.1. Cone programs. We consider the cone problems of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \langle c, x \rangle \quad \text{s.t.} \quad Ax + s = b, \quad s \in \mathcal{K} \quad (7.1)$$

where \mathcal{K} is a nonempty closed convex cone. Almost any convex program can be recast as (7.1), and many convex optimization solvers address problems by first translating them into this form. The KKT conditions for optimality of the primal-dual couple $((x^*, s^*), (y^*, r^*))$ are

$$Ax^* + s^* = b, \quad s^* \in \mathcal{K}, \quad A^T y^* + c = r^*, \quad r^* = 0, \quad y^* \in K^*, \quad (y^*)^T s^* = 0$$

where K^* is the dual cone of \mathcal{K} . A recently developed conic solver for (7.1) is SCS [18], which solves the corresponding so-called *homogeneous self-dual embedding*, namely the variational inequality

$$\text{find } u = (x, y, \tau) \quad \text{s.t.} \quad 0 \in Qu + N_{\mathcal{C}}(u) \quad (7.2)$$

where

$$\mathcal{C} = \mathbb{R}^n \times K^* \times \mathbb{R}_+ \quad \text{and} \quad Q = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix}.$$

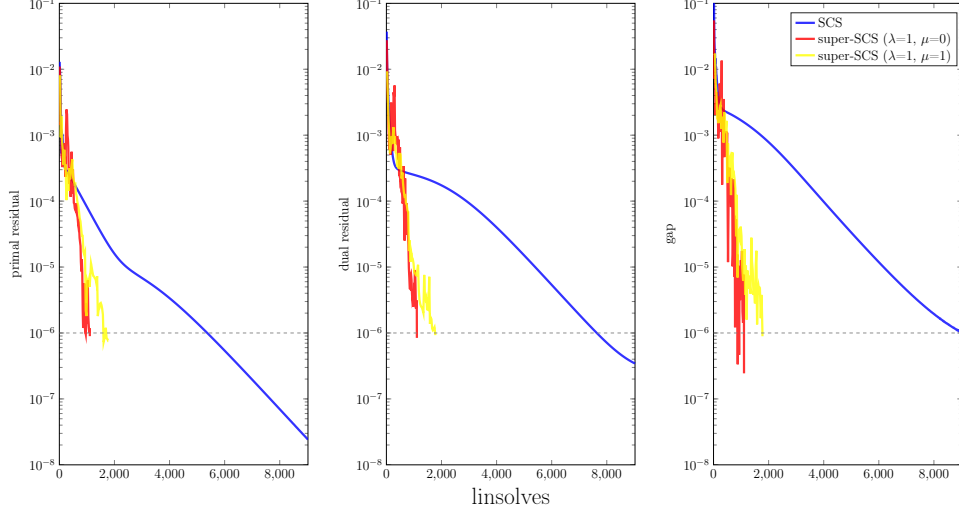
As a short and elegant interpretation of the method, SCS basically tackles (7.2) by means of Douglas-Rachford splitting (DRS), namely

$$\begin{cases} \tilde{u} & \approx (I + \gamma Q)^{-1}(u) \\ \bar{u} & = \Pi_{\mathcal{C}}(2\tilde{u} - u) \\ u^+ & = u + \bar{u} - \tilde{u} \end{cases} \quad (7.3)$$

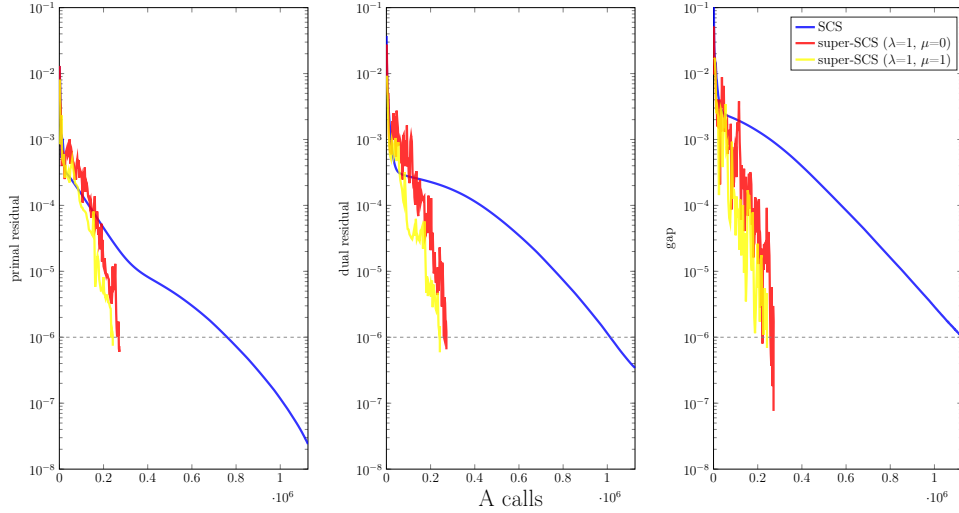
The “ \approx ” symbol refers to the fact that \tilde{u} may be retrieved inexactly by means of conjugate gradient method; we refer to [18] for a detailed discussion.

As anticipated in the Introduction, DRS is a nonexpansive operator and as such it can be integrated in the *SuperMann scheme*. In fact, it is *firmly* nonexpansive, so that any $\lambda \in (0, 2)$ can be selected; in these simulations we set $\lambda = 1$. We run a cone problem (7.1) of size $m = 487$ and $n = 325$, with $\text{dens}(A) = 0.01$ and $\text{cond}(A) = 100$, both by solving exactly the linear systems and by adopting the CG technique. We also compare the choice $\mu = 0$ with $\mu = \lambda$ so to show that even if each iteration is more expensive in the latter case due to the extra evaluation of T , in terms of number of operations such a choice might be beneficial. We reported primal residual, dual residual, and duality gap; consistently with SCS' termination criterion, the algorithm is stopped when all these quantities are below some tolerance (cf. [18, §3.5]), which we set to 10^{-6} .

FIGURE 4. Comparison between Splitting Cone Solver [18] (blue) and its enhancement with the *SuperMann scheme* for solving a cone program (7.1): in red without the extra nominal step (i.e., $\mu = 0$), and in yellow with $\mu = \lambda$.



(A) On the x -axis the number of times a linear system is solved, the most expensive operation, needed for computing the resolvent of Q . SCS performs quite well, however its super-enhancement converges considerably faster in terms of operations.



(B) Comparison with respect to the same problem, but with linear systems solved approximately with CG on a reduced system. On the x -axis the number of times either the operator A or A^* is called, which amount to the most expensive operations. The comparison between SCS and super-SCS is quite identical to that in which the linear system is solved exactly. Differently from the exact simulation in Figure 5a, this time operating the extra nominal step in *superSCS* seems to perform slightly better.

In Figure 4 we can observe how the original SCS scheme (blue) converges at a fair linear rate; however, its super-enhancement greatly outperforms it both when solving linear system exactly and approximately.

7.2. Lasso. We consider a lasso problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \frac{1}{2} \|Ax - b\|^2 + \nu \|x\|_1$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\nu > 0$. In Figure 6 the comparison of forward-backward splitting (or proximal gradient, in blue) and its super-enhanced version (red) with $\mu = 0$ (i.e., no without extra nominal step) on a random problem with $m = 1500$ $n = 5000$ and $\nu = 10^{-2}$. On the x -axis the number of matvecs, being them the most expensive operations of FB and hence of super-FB, and on the y -axis the fixed-point residual. Though superlinear convergence cannot be observed due to the fact that a limited-memory method is used for computing directions, however an outstanding speedup is noticeable.

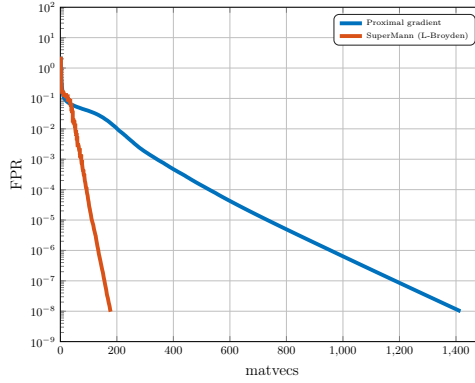


FIGURE 6. Comparison between FBS and Super-FBS (using modified Broyden limited-memory directions) in a lasso problem.

8. CONCLUSIONS

We proposed the *SuperMann scheme* (Alg. 3), a novel algorithm for finding fixed points of a nonexpansive operator T that generalizes and greatly improves the classical Krasnosel'skiĭ-Mann (KM) scheme, enjoying the same favourable properties, namely: global (weak) convergence with worst-case sublinear rate, cheap iterations based solely on evaluations of T and (possibly) matrix-vector products, and easy codability. The *SuperMann scheme* is an extremely versatile algorithm, its flexibility being twofold: on one hand it works for any nonexpansive operator T by requiring only the oracle $x \mapsto Tx$; on the other hand it allows for the integration of any fast local method for solving nonlinear equations, leaving much freedom for trading-off cheap iterations or faster convergence. The remarkable performance of the method is supported both in practice with promising simulations and in theory where the employment of limited-memory quasi-Newton directions is shown to yield asymptotic superlinear convergence rates provided a condition analogous to the famous result by Dennis and Moré is satisfied.

We encourage the employment of the *SuperMann scheme* to greatly improve and robustify convex splitting algorithms; for the same reasons we strongly believe that its integration in convex solvers which are based on fixed-point iterations of nonexpansive operators such as SCS would be extremely beneficial.

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APPENDIX A. PROOFS OF SECTION 6

We first prove a preliminary result.

Lemma A.1. *Starting from $B_0 \in \mathcal{B}(\mathcal{H})$, let $(B_k)_{k \in \mathbb{N}}$ be defined as the relaxed Broyden's update (6.7) with respect to some $(\theta_k)_{k \in \mathbb{N}} \subset [0, 2]$. Then, for any $G \in \mathcal{B}(\mathcal{H})$ and $k \in \mathbb{N}$ it holds that*

$$\|B_{k+1} - G\|_{\mathcal{B}} \leq \|B_k - G\|_{\mathcal{B}} - \frac{\theta_k(2 - \theta_k)}{2\|B_k - G\|_{\mathcal{B}}} \frac{\|(B_k - G)s_k\|^2}{\|s_k\|^2} + \theta_k \frac{\|y_k - Gs_k\|}{\|s_k\|}$$

where the second term in the right-hand side is by convention null if $B_k = G$.

Proof. Let $E_k := B_k - G$. From (6.9) we have

$$E_{k+1} = \Pi_{S_k, \theta_k} B_k - G = (\Pi_{S_k, \theta_k} B_k - \Pi_{S_k, \theta_k} G) - \theta_k(\text{Id} - \Pi_{S_k})G \quad (\text{A.1})$$

Applying (2.2) on the term between in round brackets we obtain

$$\begin{aligned} \|\Pi_{S_k, \theta_k} B_k - \Pi_{S_k, \theta_k} G\|_{\mathcal{B}}^2 &\leq \|E_k\|_{\mathcal{B}}^2 - \theta_k(2 - \theta_k) \|(\text{Id} - \Pi_{S_k})B_k - (\text{Id} - \Pi_{S_k})G\|_{\mathcal{B}}^2 \\ &\stackrel{(6.4)}{=} \|E_k\|_{\mathcal{B}}^2 - \theta_k(2 - \theta_k) \frac{\|E_k s_k \otimes s_k\|_{\mathcal{B}}^2}{\|s_k\|^4} \\ &\stackrel{(2.1)}{=} \|E_k\|_{\mathcal{B}}^2 - \theta_k(2 - \theta_k) \frac{\|E_k s_k\|^2}{\|s_k\|^2} \end{aligned}$$

Using the inequality $\sqrt{\alpha^2 - \beta^2} \leq \alpha - \beta^2/2\alpha$ which holds for any α, β satisfying $0 \neq \alpha \geq |\beta| \geq 0$, we then obtain

$$\|\Pi_{S_k, \theta_k} B_k - \Pi_{S_k, \theta_k} G\|_{\mathcal{B}} \leq \|E_k\|_{\mathcal{B}} - \frac{\theta_k(2 - \theta_k)}{2\|E_k\|_{\mathcal{B}}} \frac{\|E_k s_k\|^2}{\|s_k\|^2} \quad (\text{A.2})$$

Moreover,

$$\|(\text{Id} - \Pi_{S_k})G\|_{\mathcal{B}} \stackrel{(6.4)}{=} \left\| \frac{y_k - Gs_k}{\|s_k\|^2} \otimes s_k \right\|_{\mathcal{B}} \stackrel{(2.1)}{=} \frac{\|y_k - Gs_k\|}{\|s_k\|} \quad (\text{A.3})$$

The triangular inequality on (A.1) combined with (A.2) and (A.3) proves the claimed result. \square

We now have all the needed material for the convergence proof of the modified Broyden's method.

Proof of Theorem 6.3.

Suppose that $F(x^k) \neq 0$ for all k , and let L_F be the calmness modulus of the semiderivative of F at x^* in a neighborhood U_{x^*} . Then,

$$\begin{aligned} \frac{\|y_k - G_{\star} s_k\|}{\|s_k\|} &= \frac{\|F(x^{k+1}) - F(x^k) - G_{\star}(x^{k+1} - x^k)\|}{\|x^{k+1} - x^k\|} \\ &\leq L_F \max \{ \|x^k - x^*\|, \|x^{k+1} - x^*\| \} \end{aligned}$$

where the last inequality follows from Lem. 2.12. In particular, in light of the assumption on $(\|x^k - x^*\|)_{k \in \mathbb{N}}$,

$$\sum_{k \in \mathbb{N}} \frac{\|y_k - G_{\star} s_k\|}{\|s_k\|} < \infty.$$

Let $E_k = B_k - G_{\star}$; then, Lem. A.1 yields

$$\|E_{k+1}\|_{\mathcal{B}} \leq \|E_k\|_{\mathcal{B}} - \frac{\theta_k(2 - \theta_k)}{2\|E_k\|_{\mathcal{B}}} \left(\frac{\|(B_k - G_{\star})s_k\|}{\|s_k\|} \right)^2 + \theta_k \frac{\|y_k - G_{\star} s_k\|}{\|s_k\|}$$

with the same convention of the second term being zero if $E_k = 0$. The last term on the right-hand side, be it σ_k , is summable and therefore the sequence $(E_k)_{k \in \mathbb{N}}$

is bounded. Let $\bar{E} = \mathbf{max}(\|E_k\|_{\mathcal{B}})_{k \in \mathbb{N}}$ and notice that from the assumptions it follows that $\theta_k(2 - \theta_k) \geq \bar{\theta}(2 - \bar{\theta}) > 0$, yielding

$$\|E_{k+1}\|_{\mathcal{B}} - \|E_k\|_{\mathcal{B}} \leq \sigma_k - \frac{\bar{\theta}(2 - \bar{\theta})}{2\bar{E}} \left(\frac{\|(B_k - G_{\star})s_k\|}{\|s_k\|} \right)^2$$

Telescoping the above inequality, summability of σ_k ensures that of $\frac{\|(B_k - G_{\star})s_k\|^2}{\|s_k\|^2}$ proving in particular the claimed Dennis-Moré condition. \square

Proof of Theorem 6.5.

In light of [Lem. 2.8](#) there exist $\varepsilon, \kappa > 0$ such that

$$\|Rx\| = \|Rx - Rx^{\star}\| \geq \kappa\|x - x^{\star}\| \quad \forall x \in B(x^{\star}, \varepsilon). \quad (\text{A.4})$$

In particular, by [Lem. 2.3\(i\)](#) it follows that $\mathbf{fix} T = \{x^{\star}\}$, and therefore

$$\frac{1}{\kappa}\|Rx\| \geq \|x - x^{\star}\| = \mathbf{dist}(x, \mathbf{fix} T) \quad \forall x \in B(x^{\star}, \varepsilon).$$

This shows metrical subregularity of R at x^{\star} , hence the summability of $(\|Rx^k\|)_{k \in \mathbb{N}}$ due to [Thm. 5.1\(vi\)](#), and in turn that of $(\|x^k - x^{\star}\|)_{k \in \mathbb{N}}$, as it is evident from (A.4).

The specific modified Broyden's update ensures that all B_k are invertible; moreover,

$$\begin{aligned} \|w^k - x^{\star}\| &= \|T_{\mu}x^k - x^{\star} + \tau_k B_k^{-1} R T_{\mu}x^k\| \leq \|T_{\mu}x^k - x^{\star}\| + \|B_k^{-1}\| \|R T_{\mu}x^k\| \\ &\stackrel{(\text{A.4})}{\leq} (1/\kappa + \|B_k^{-1}\|) \|R T_{\mu}x^k\| \leq (1/\kappa + \|B_k^{-1}\|) \|Rx^k\| \end{aligned}$$

where the last inequality follows from [Lem. 2.3\(v\)](#). By boundedness of $(B_k^{-1})_{k \in \mathbb{N}}$ and since $(\|Rx^k\|)_{k \in \mathbb{N}} \in \ell^1$ it follows that $\|w^k - x^{\star}\| \in \ell^1$ too. Due to [Rem. 6.4](#) we have that [Thm. 6.3](#) applies, and since $d^k = -B_k^{-1}Rx^k$ and $s_k = -\tau_k d^k$ we have

$$0 \leftarrow \lim_{k \rightarrow \infty} \frac{\|(B_k - G_{\star})s_k\|}{\|s_k\|} = \lim_{k \rightarrow \infty} \frac{\|(B_k - G_{\star})d_k\|}{\|d_k\|} = \lim_{k \rightarrow \infty} \frac{\|Rx^k + G_{\star}d_k\|}{\|d_k\|}$$

as claimed. \square